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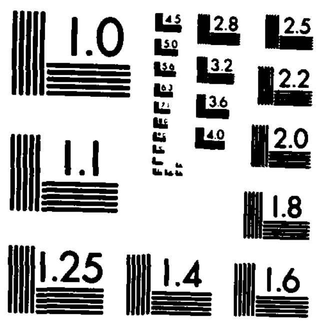

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EXISTENCE OF FORCED OSCILLATIONS FOR SOME NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT

This article studies the existence of T-periodic solutions for systems of nonlinear second order ordinary differential equations of the type $\ddot{x} + V'(x) = f(t)$. Here, $x : R \to R^N$, $V \in C^2(R^N, R)$ and $f : R \to R^N$ is a given T-periodic forcing term ($T > 0$ is given). Assuming function V to be superquadratic, it is shown that this system possesses infinitely many T-periodic solutions. The proof of this result rests on showing that certain homotopy groups of level sets of the functional associated with the system are not trivial. Some more general results concerning systems of ~~the~~ type $\ddot{x} + \hat{V}'_x(t,x) = 0$ are also presented here.

AMS (MOS) Subject Classifications: Primary: 34C15, 58F05; Secondary: 34C25, 58E05

Key Words: Second order system of ordinary differential equations, Nonlinear forced oscillations, Periodic solutions, S^1-action, Critical points, Level sets, Homotopy groups of level sets

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SIGNIFICANCE AND EXPLANATION

Systems of the type $\ddot{x} + V'(x) = 0$ (where $x = x(t) \in \mathbb{R}^N$ and $V \in C^1(\mathbb{R}^N, \mathbb{R})$) describe the motion of a mechanical system consisting of a finite number of points $x_1, \ldots, x_N$, with a potential given by the function $V(x_1, \ldots, x_N)$. In the presence of external forces, the system to be studied is:

$$(*) \qquad \ddot{x} + V'(x) = f(t) .$$

Assuming that the forcing term $f(t)$ is T-periodic in time, one would like to know whether (*) has a T-periodic response. Under the assumption that V has superquadratic growth as $|x| \to +\infty$, it is shown in this paper that the answer is affirmative; in fact, (*) has infinitely many T-periodic vibrations induced by the forcing term f.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

EXISTENCE OF FORCED OSCILLATIONS FOR
SOME NONLINEAR DIFFERENTIAL EQUATIONS

Abbas Bahri* and Henri Berestycki**

1. INTRODUCTION AND MAIN RESULTS

This paper is concerned with the existence of T-periodic solutions $(T \in \mathbb{R}, T > 0$ given) for the following second order system of nonlinear ordinary differential equations:

$$\ddot{x} + V'(x) = f(t) \ . \tag{1.1}$$

Here, $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$, $x : \mathbb{R} \to \mathbb{R}^N$, $V \in C^1(\mathbb{R}^N, \mathbb{R})$, $V'(x)$ is the gradient of V and $f : \mathbb{R} \to \mathbb{R}^N$ is some given T-periodic "forcing" term. The main purpose of this paper is to show that if $V(x)$ is superquadratic as $|x| \to +\infty$, then (1.1) possesses infinitely many T-periodic solutions ("nonlinear forced oscillations").

More precisely, we assume that V satisfies the following condition:

$$\text{(V)} \qquad \begin{cases} 0 < V(x) \leq \theta\, V'(x)\cdot x \quad \text{for all} \quad x \in \mathbb{R}^N, \quad |x| \geq R\ , \\ \text{with} \quad 0 < \theta < \frac{1}{2}\ , \quad \text{for some} \quad R > 0\ . \end{cases}$$

(*Here,* $V'(x)\cdot x$ *denotes* the scalar product in $\mathbb{R}^N$). From (V) via an integration it is easily derived that V is superquadratic at infinity; that is, V satisfies:

$$\frac{a}{p+1}|x|^{p+1} - b \leq V(x), \qquad \forall\, x \in \mathbb{R}^N\ , \tag{1.2}$$

with $p + 1 = \frac{1}{\theta} > 2$ and $a,b > 0$ being constants.

Let us now state our main result.

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Theorem 1. Suppose that $V \in C^2(\mathbb{R}^N,\mathbb{R})$ satisfies condition (V). Then, for any given $f \in L^2_{loc}(\mathbb{R},\mathbb{R}^N)$ which is T-periodic, the system (1.1) admits infinitely many T-periodic solutions[(1)].

The proof of this result will take up sections 2 to 6. In Section 7, the same method is applied to obtain the existence of periodic solutions for more general non autonomous systems of the type

$$\ddot{x} + \hat{V}'_x(t,x) = 0 \ . \tag{1.3}$$

There is a vast literature devoted to the subject of nonlinear oscillations in systems like (1.1) or (1.3). However, in the case of a superquadratic V, for a system (1.1), even the existence of at least one periodic solution for any given periodic f was an open problem. Let us recall some previous works in this domain.

Firstly, in the case of a single scalar equation $(N = 1)$:

$$\ddot{x} + g(t,x) = 0 \qquad (x(t) \in \mathbb{R}) \ , \tag{1.4}$$

quite general results on the existence of periodic solutions have been obtained by Hartman [14] and Jacobowitz [15] (by using the Poincaré-Birkhoff Theorem). For earlier works in this case $N = 1$, the reader is also referred to Cesari [10], Ehrmann [11], Micheletti [17], Fučik and Lovicar [13], Nehari [18] and Wolkowiski [26]. (See also the book by S. Fučik [12, Chapter 36] which mentions the open problem of extending the results from scalar equations to systems).

For systems, when $N \geq 2$, existence of free oscillations in the autonomous system

$$\ddot{x} + V'(x) = 0 \tag{1.5}$$

(i.e. $f \equiv 0$ in (1.1)) have been established for $V \in C^1(\mathbb{R}^N,\mathbb{R})$ satisfying condition (V) by Benci [7] and Rabinowitz [20, 23]. The methods they use rely on the autonomous character of (1.5) (or equivalently, on the S^1-invariance of the associated functional -

---

(1) A weaker version of this result was announced in our Note [5] where an additional assumption was imposed on V; in particular V was restricted to have at most polynomial growth at infinity.

see below) and do not apply readily for a forced system like (1.1). As a first step in the proof of Theorem 1, we will derive the result concerning free oscillations by a new and somewhat simpler proof[(1)].

The present paper is, in a sense, a continuation of [6]. There, we studied the existence of forced oscillations for Hamiltonian systems of the type

$$(1.6)\qquad \begin{aligned} \dot{x} &= -\frac{\partial H}{\partial p}(x,p) + f_1(t) \\ \dot{p} &= \frac{\partial H}{\partial x}(x,p) + f_2(t) . \end{aligned}$$

In (1.6), $z = (x,p) : \mathbb{R} \to \mathbb{R}^{2N}$, $H(z) \in C^2(\mathbb{R}^{2N},\mathbb{R})$; $(f_1,f_2) : \mathbb{R} \to \mathbb{R}^{2N}$ is given, of class C^1 and T-periodic. In [6], H was assumed to satisfy the same condition as (V) (with respect to the variable $z = (x,p)$). H was furthermore required to verify:

$$(1.7)\qquad \begin{aligned} &a|z|^{p+1} - b \leq H(z) \leq a'|z|^{q+1} + b' \quad \forall\, z \in \mathbb{R}^{2N} , \\ &\text{with} \\ &1 < p \leq q < 2p+1 \quad \text{and} \quad a,b,a',b' > 0 . \end{aligned}$$

Under these conditions, we derived in [6] the existence of infinitely many T-periodic solutions of (1.6).

Now (1.1) is but a particular case of a Hamiltonian system like (1.6), of special importance in mechanics. Indeed, (1.1) corresponds to a separable Hamiltonian

$$(1.8)\qquad H(x,p) = \frac{1}{2}|p|^2 + V(x)$$

and $f = \dot{f}_1 - f_2$. Theorem 1, however, is not contained in the results of [6]. Firstly, from (1.7) one sees that the Hamiltonian H corresponding to (1.1) is not superquadratic in both variables x and p[(2)]. Furthermore, it should be emphasized that on the contrary of the results of [6] about (1.6), no additional assumption to (V) (e.g. like (1.7)) is being imposed here on V.

---

(1) Very recently, and independently, Rabinowitz [24] has proved a weaker version of Theorem 1 under an additional growth restriction on V. The approach used in [24] is different from albeit not unrelated to ours.

(2) Existence results for the system (1.6) that would contain both the case of a superquadratic H and (1.8) are still by and large open.

The structure of the proof of Theorem 1 parallels the ideas we developed in [6]. But the framework and, chiefly, some crucial estimates and the way these are established are quite different for the two problems. Therefore, we have separated the study of (1.1) from the results concerning (1.6). We shall nevertheless use here a few results from [6] without repeating the proofs. The methods of the present paper are also to be compared with the ones we used in [3, 4] to study some superlinear elliptic partial differential equations(1).

In this paper, as in [6], we will use the recent work of A. Bahri [1, 2] in Morse theory which concerns the relationship between critical points of a functional and homotopy groups of its level sets. In Section 2 we state in an abstract setting and recall the proof of the precise result that will be used in the sequel.

To prove Theorem 1, we first construct in Section 3 a sequence of critical values $(c_k)_{k\in\mathbb{N}}$ for the autonomous system (1.5). The level sets of the functional associated with (1.5) corresponding to the numbers c_k are then shown to have some topological property which, in some sense, is stable under perturbations. We also require a sharp estimate from below on the growth of the c_k as $k \to +\infty$. This is obtained by carefully analyzing in Section 4 a certain autonomous equation that serves for comparison purposes. We conclude in Section 6 by using a perturbation argument on the autonomous functional which allows us to find periodic solutions of (1.1).

In the last section, we study more general perturbations from an autonomous system of the type (1.3). There, we derive some results about the existence of infinitely many periodic solutions of (1.3) which extend Theorem 1.

This paper is thus organized as follows:

1. Introduction and main results.
2. A theorem on the homotopy groups of level sets of a functional.
3. Critical values and periodic solutions in the autonomous case.

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(1)Actually, the methods of the present paper could also be used to slightly improve the results of [3, 4].

4. A detailed study of some autonomous equation.

5. An estimate from below on the growth of the critical values.

6. Existence of forced oscillations.

7. More general forced systems.

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2. A THEOREM ON THE HOMOTOPY GROUPS OF LEVEL SETS OF A FUNCTIONAL

In the course of the proof of Theorem 1 as well as in the last section, we will use a result concerning the relationship between certain homotopy groups of level sets of a functional and its critical points. The main idea is to adapt a classical theorem from Morse theory to situations which may be "degenerate". This adaptation relies on an approximation procedure of Marino and Prodi [16]. In this section we state in an abstract setting and recall the proof of the precise theorem that will be required thereafter. This result is due to A Bahri [1, 2] and we refer the reader to [1,2] for more general properties in this direction.

We start by recalling the following fact from Morse theory. Let M be a smooth Hilbert manifold. Let $f \in C^2(M,\mathbb{R})$ satisfy the Palais-Smale condition (see below); we denote $M_a = \{x \in M,\ f(x) \geq a\}$. Let $b < a$ be two given reals which are regular values of f. Assume that the set $Z_b^a(f) = \{x \in M,\ b \leq f(x) \leq a,\ f'(x) = 0\}$ is finite and that $\forall\, x \in Z_b^a(f)$, x is a nondegenerate critical point of f (i.e. the Hessian form $f''(x)$ is definite). Recall that the coindex of x is the maximum dimension of a subspace of T_xM on which $f''(x)$ is positive definite. Then, one has the following result (Theorem 7.3 in J. T. Schwartz [25]).

<u>Proposition 2.1.</u> In addition to the above hypotheses, assume that for any $x \in Z_b^a(f)$, the coindex of x is larger than n. Then $\pi_n(M_b, M_a) = 0$.

Here $\pi_n(M_b,M_a)$ denotes the relative homotopy group of the pair M_b, M_a; $n \in \mathbb{N}^* = \mathbb{N}\backslash\{0\}$. From now on, let $M = H$ be a finite dimensional Hilbert space. Let $f \in C^2(H,\mathbb{R})$ and assume that f satisfies the following Palais-Smale condition.

$$(\text{P.S})_1 \qquad \begin{cases} \text{For any sequence } (x_j) \subset H \text{ such that } f(x_j) \\ \text{is bounded and } f'(x_j) \to 0, \text{ there exists} \\ \text{a convergent subsequence from } (x_j). \end{cases}$$

We denote $Z^a(f) = \{x \in H;\ f'(x) = 0,\ f(x) < a\}$ and $[f]_a = \{x \in H;\ f(x) > a\}$. From the previous proposition we derive:

Proposition 2.2. Let $f \in C^2(H,\mathbb{R})$ verify condition $(\text{P.S})_1$. Suppose that for some regular value of f, $a \in \mathbb{R}$, $Z^a(f)$ is finite and that for any $x \in Z^a(f)$, x is non-degenerate and has coindex larger than n. (That is, $f''(x)$ has at least $n+1$ positive eigenvalues, counting multiplicities, and $f''(x)$ does not have 0 as an eigenvalue). Then, $\pi_\ell([f]_a,p) = 0 \quad \forall\, \ell \le n-1,\ \ell \in \mathbb{N}^*,\ \forall\, p \in [f]_a$.

Here, $\pi_\ell([f]_a,p)$ denotes the (absolute) homotopy group of order ℓ of $[f]_a$ with base point p. To prove this proposition, we require the following well known lemma ("non-critical neck principle").

Lemma 2.1. Let $f \in C^1(H,\mathbb{R})$ verify condition $(\text{P.S})_1$. Let $b \in \mathbb{R}$ be such that f has no critical values in $(-\infty,b]$. Then, $[f]_b$ is a deformation retract of H.

Proof of Lemma 2.1. Firstly, by $(\text{P.S})_1$, there exists $b_1 > b$ such that f has no critical values in $(-\infty,b_1]$. Let $\rho : H \to \mathbb{R}$ be a locally Lipschitz function such that $0 \le \rho \le 1$, $\rho \equiv 1$ on the set $\{x \in H;\ f(x) \le b\}$ and $\rho \equiv 0$ on $[f]_{b_1}$. (Such a function is easily constructed explicitly; see e.g. Rabinowitz [22]). Let v denote a "pseudo-gradient vector field" for f on the set $[f]^{b_1} = \{x \in H;\ f(x) \le b_1\}$. That is, $v : [f]^{b_1} \to H$ is a locally Lipschitz mapping satisfying:

$$\langle f'(x), v(x)\rangle \ge \|f'(x)\|^2 ,$$

$$\|v(x)\| \le 2\|f'(x)\|$$

for all x in $[f]^{b_1}$ where $\langle\ ,\ \rangle$ and $\|\ \|$ denote the scalar product and norm in H. The existence of such a vector field under the assumptions of Lemma 2.1 is classical (see e.g. [22]).

Consider the initial problem

$$\frac{d\eta}{dt} = \rho(\eta)\,\frac{v(\eta)}{\|v(\eta)\|^2} \qquad \eta(0,x) = x \tag{2.1}$$

$(\eta = \eta(t,x))$. Using condition $(P.S)_1$ it is easily verified that (2.1) has a unique solution $\eta(t,x)$ defined for all $t \in \mathbb{R}$ and $x \in H$, and that for each $t \in \mathbb{R}$, $x \to \eta(t,x)$ is a homeomorphism: $H \to H$. Clearly, if $x \in [f]_{b_1}$, then $\eta(t,x) = x\ \forall\ t \in \mathbb{R}$. One also has:

$$\frac{df(\eta)}{dt} = \rho(\eta)\,\frac{\langle f'(\eta),v(\eta)\rangle}{\|v(\eta)\|^2} \geq \frac{1}{4}\,\rho(\eta) \geq 0\ . \tag{2.2}$$

Hence, the function $t \to f(\eta(t,x))$ is nondecreasing. If $f(\eta(t,x)) < b$ for some $t > 0$, then $f(\eta(s,x)) < b$ whence $\rho(\eta(s,x)) = 1$ for all $s \in [0,t]$. In this case, therefore, (2.2) implies

$$f(\eta(t,x)) - f(x) \geq \frac{1}{4}\,t\ .$$

Denote $c^+ = \max(c,0)$ for $c \in \mathbb{R}$ and let

$$r(t,x) = \eta[4t(b - f(x))^+,x],\quad t \in [0,1],\quad x \in H\ .$$

Then, $r : [0,1] \times H \to H$ is continuous, $r(t,x) = \eta(0,x) = x$ for all $x \in [f]_b$, $r(0,x) = x$, $\forall\ x \in H$, and, lastly, $r(1,x) \in [f]_b$, $\forall\ x \in H$. Thus, $[f]_b$ is a deformation retract of H.

<u>Proof of Proposition 2.2.</u> Let $b \in \mathbb{R}$, $b < a$ be such that f has no critical values in $(-\infty,b]$. Since $[f]_b$ is a retract of H, one has

$$\pi_\ell([f]_b,p) = 0,\quad \forall\ \ell \in \mathbb{N}^*,\quad \forall\ p \in [f]_b\ , \tag{2.3}$$

By Proposition 2.1, one knows that

$$\pi_\ell([f]_b,[f]_a) = 0,\quad \forall\ \ell \in \mathbb{N}^*,\quad \ell < n\ . \tag{2.4}$$

For $p \in [f]_a$, one has the exact sequence:

(2.5) $$\cdots \to \pi_{\ell+1}([f]_b,[f]_a) \xrightarrow{\partial} \pi_\ell([f]_a,p) \xrightarrow{i^*} \pi_\ell([f]_b,p) \to \pi_\ell([f]_b,[f]_a) \to \cdots$$

Using (2.3) and (2.4), the exact sequence (2.5) yields

(2.6) $$0 \to \pi_\ell([f]_a,p) \to 0, \quad \forall\, \ell \in \mathbb{N}^*, \quad \ell \le n-1 .$$

The proof of Proposition 2.2 is thereby complete. ∎

The setting of Proposition 2.2 is "nondegenerate" in the sense of Morse theory. That is, $Z^a(f)$ is finite and any $x \in Z^a(f)$ is assumed to be nondegenerate. The main result of this section is the following theorem (A. Bahri [1, 2]) which extends Proposition 2.2 to situations which may be degenerate in the above sense.

Theorem 2. Let H be a finite dimensional Hilbert space. Let $f \in C^2(H,\mathbb{R})$ be a functional satisfying condition $(P.S)_1$. Assume that a is not a critical value of f and that $Z^a(f) = \{x \in H;\ f(x) < a,\ f'(x) = 0\}$ is compact. Suppose furthermore that for any $x \in Z^a(f)$, there exists a subspace $H_x \subset H$ such that $\dim H_x > n$ and $f''(x)$ is a positive definite bilinear form on H_x (i.e. $f''(x)$ has at least $n+1$ positive eigenvalues). Then,

$$\pi_\ell([f]_a,p) = 0 \qquad \forall\, \ell \in \mathbb{N}^*,\ \ \ell \le n-1,\ \ \forall\, p \in [f]_a .$$

The proof of Theorem 2 rests on the following approximation result of Marino and Prodi [16] (see also Proposition 6.2 in [6]).

Proposition 2.3. Let Ω be a C^2 open subset of some Hilbert space $\mathcal{H}$ and let $\phi \in C^2(\Omega,\mathbb{R})$. Assume that ϕ' is a Fredholm operator (hence of null index) on the critical set $Z(\phi) = \{x \in \Omega;\ \phi'(x) = 0\}$. Lastly, suppose that ϕ verifies the condition $(P.S)_1$ and that $Z(\phi)$ is compact. Then, for any $\varepsilon_0 > 0$ and $\eta_0 > 0$, there exists $\psi \in C^2(\Omega,\mathbb{R})$ verifying $(P.S)_1$ with the following properties.

i) $\psi(u) = \phi(u)$ if $\mathrm{distance}\{u, Z(\phi)\} > \eta_0$

ii) $\|\psi(u) - \phi(u)\|,\ \|\psi'(u) - \phi'(u)\|,\ \|\psi''(u) - \phi''(u)\| < \varepsilon_0, \quad \forall\, u \in \Omega$

iii) The critical points of ψ (if any) are in finite number and nondegenerate.

Remark 2.1. This result is proved in [16]. The only modification with respect to the statement in Marino-Prodi [16] concerns property ii) where we have added the requirement

$\|\psi''(u) - \phi''(u)\| < \varepsilon_0$. However, an inspection of the proof of [16] readily shows that this condition can be fulfilled as well by the very same construction. ∎

Proof of Theorem 2. For $x \in H$, $r,\alpha \in \mathbb{R}$, $r,\alpha > 0$ and $A \quad H$ we denote $B(x,r) = \{y \in H;\ \|y - x\| < r\}$, and $N_\alpha(A) = \{x \in H;\ \text{distance}\{x,A\} < \alpha\}$. Firstly, let us remark that since H is finite dimensional, there exists $\varepsilon_x > 0$ such that

$$\langle f''(x)h,h\rangle \geq \varepsilon_x \|h\|^2 \qquad \forall\, h \in H_x . \tag{2.7}$$

Since f is of class C^2, there exists a ball $B(x,r_x)$ centered at x of radius $r_x > 0$ such that

$$\langle f''(y)h,h\rangle \geq \frac{\varepsilon_x}{2} \|h\|^2 \qquad \forall\, h \in H_x,\ \forall\, y \in B(x,r_x) . \tag{2.8}$$

Let $x_1,\dots,x_p \in Z^a(f)$ be such that $B(x_1,r_{x_1}),\dots,B(x_p,r_{x_p})$ form a covering of $Z^a(f)$. Let $\eta_1 > 0$ be such that

$$N_{\eta_1}(Z^a(f)) \subset \bigcup_{j=1}^{p} B(x_j,r_{x_j}) .$$

Let us now apply Proposition 2.3 with $\mathcal{H} = H$, $\phi = f$, and $\Omega = \{x \in H;\ f(x) < a\}$. Since a is not a critical value of f and $Z_a(f)$ is compact, Ω is a C^2-open subset of H and there exists η_2 such that $\text{distance}\{x, Z^a(f)\} < \eta_2$ implies $f(x) < a$.

Let $\eta_0 = \min\{\eta_1,\eta_2\} > 0$. Lastly, we choose $\varepsilon_0 > 0$ such that

$$\varepsilon_0 < \frac{1}{4}\min\{\varepsilon_{x_1},\dots,\varepsilon_{x_p}\} \tag{2.9}$$

and

$$\varepsilon_0 < a - \max_{x \in N_{\eta_0}(Z^a(f))} f(x) \tag{2.10}$$

Then, by Proposition 2.3, there exists $\psi \in C^2(\Omega,\mathbb{R})$ verifying i)-iii). Let $g(x) = \psi(x)$ if $x \in \Omega$ and $g(x) = f(x)$ if $f(x) \geq a$. Noticing that there is some $\varepsilon > 0$ such that $\psi(x) = f(x)$ for any $x \in \Omega$ with $f(x) \geq a - \varepsilon$, it is readily seen that $g \in C^2(H, \mathbb{R})$. Furthermore, by i), ii) and (2.10) one has $[g]_a = [f]_a$.

Since $Z^a(g) \subset N_{\eta_0}(Z^a(f))$, for any $y \in Z^a(g)$ there exists $j \in \{1,\dots,p\}$ such that $y \in B(x,r_x)$, with $x = x_j$. Hence, using (2.8) and the fact that

$$\|f''(y) - g''(y)\| \leq \frac{1}{4}\varepsilon_x ,$$

one obtains

$$\langle g''(y)h,h\rangle \geq \frac{\varepsilon_x}{4}\|h\|^2 \quad \forall\, h \in H_x.$$

Therefore, as $\dim H_x > n$, the coindex of y is larger than n, for all $y \in Z^a(g)$. By Proposition 2.2 one then has $\pi_\ell([g]_a, p) = 0$, $\forall\, \ell \in \mathbb{N}^*$, $\ell \leq n-1$, $\forall\, p \in [g]_a$.

Since $[g]_a = [f]_a$, the proof of Theorem 2 is thereby complete. ∎

<u>Remark 2.2.</u> The compactness hypothesis on $Z^a(f)$ in Theorem 2 is certainly verified if f satisfies the following stronger Palais-Smale condition:

$$(\text{P.S}) \qquad \begin{cases} \text{For any sequence } (x_j) \subset H \text{ such that } f(x_j) \leq C \\ \text{(for some } C \in \mathbb{R}) \text{ and } f'(x_j) \to 0, \text{ there exists} \\ \text{a convergent subsequence from } (x_j). \end{cases}$$

The functionals that we will consider in the sequel do satisfy this stronger version. ∎

3. CRITICAL VALUES AND PERIODIC SOLUTIONS IN THE AUTONOMOUS CASE

In this section we construct critical values for the autonomous problem (1.5) and study some of their properties. In particular, this construction will allow us to prove the existence of free oscillations in (1.5) for any $V \in C^1(\mathbb{R}^N, \mathbb{R})$ satisfying condition (V). We start by setting the functional framework that we will use throughout the paper.

Without loss of generality we may assume by means of a scale change in time that $T = 2\pi$. In the following, as is customary, 2π-periodic functions will be thought of as defined on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Let $E = (H^1(S^1))^N$. E is endowed with the Hilbert norm

$$\|x\|^2 = \int_0^{2\pi} |\dot{x}|^2 dt + \int_0^{2\pi} |x|^2 dt \qquad ^{(1)}.$$

In order to keep notations simple, we henceforth will write $H^1(S^1)$, $L^r(S^1)$ ... instead of $(H^1(S^1))^N$, $(L^r(S^1))^N$ etc... Recall that $E \hookrightarrow L^\infty(S^1)$ with a compact injection.

For $x \in E$, let

$$I^*(x) = \frac{1}{2}\int_0^{2\pi} |\dot{x}|^2 dt - \int_0^{2\pi} V(x)dt$$

---

(1) We recall that E is the space of 2π-periodic functions $x : \mathbb{R} \to \mathbb{R}^N$ such that $\|x\| < \infty$.

and

$$I(x) = \frac{1}{2}\int_0^{2\pi} |\dot{x}|^2 dt - \int_0^{2\pi} V(x)dt + \int_0^{2\pi} f\cdot x\, dt .$$

Thus, solutions of (1.1) coincide with the critical points of I in E, while the critical points of I^* in E are the 2π-periodic solutions ("free oscillations") of the autonomous system (1.5). We will also assume - without loss of generality - that $V(0) = 0$ so that $I(0) = I^*(0) = 0$.

We will now construct a sequence of critical values of I^* in E by a minimax type principle on a finite dimensional approximation of E together with a limiting procedure (Galerkin method). The spirit of this construction is to be compared with the work of Rabinowitz [19] concerning superlinear elliptic partial differential equations.

The eigenvalues of $x \to -\ddot{x}$ in E are the numbers $0,1,\ldots,m^2,\ldots$ $(m \in \mathbb{N})$. Let E^m denote the $(2m+1)N$ - dimensional subspace of E spanned by the eigenfunctions corresponding to the $(m+1)$ first eigenvalues. That is, E^m is the subspace of truncated Fourier series defined by:

$$E^m = \{x \in E;\ x(t) = \sum_{j=-m}^{+m} a_j e^{ijt},\ a_j \in \mathbb{C}^N,\ a_{-j} = \bar{a}_j,\ -m < j < m\} .$$

The group S^1 acts naturally on functions of E by time translations. For $e^{i\tau} \in S^1$ (or equivalently, $\tau \in \mathbb{R}/2\pi\mathbb{Z}$) and $x \in E$, we denote:

$$T_\tau x = x(\cdot + \tau) .$$

Clearly, the subspaces E^m are left invariant by this action $(T_\tau E^m = E^m)$ and the functional I^* is invariant:

$$I^*(T_\tau x) = I^*(x) \qquad \forall\, x \in E,\ \forall\, \tau \in \mathbb{R}/2\pi\mathbb{Z} .$$

Notice however that, in general, I is not invariant under this action.

We recall that the group S^1 acts on odd dimensional spheres. Let $k \in \mathbb{N}^*$ $(=\mathbb{N}\setminus\{0\})$ and identify $\mathbb{R}^{2k} = \mathbb{C}^k$ so that

$$S^{2k-1} = \{\zeta \in \mathbb{C}^k;\ \zeta = (\zeta_1,\ldots,\zeta_k),\ \sum_{j=1}^{k} |\zeta_j|^2 = 1\} .$$

Then, for $e^{i\tau} \in S^1$ and $\zeta \in S^{2k-1}$, we write

$$\hat{T}_\tau \zeta = e^{i\tau}\zeta = (e^{i\tau}\zeta_1,\dots,e^{i\tau}\zeta_k) .$$

A mapping $h : S^{2k-1} \to E^m$ is said to be S^1-equivariant if

$$h \circ \hat{T}_\tau = T_\tau \circ h \quad \forall\, \tau \in \mathbb{R}/2\pi\mathbb{Z} .$$

Following the same construction as in [6], we define a family of mappings and one of sets by letting, for $m > k + 1$, $m,k \in \mathbb{N}^*$:

$$\mathcal{H}_k^m = \{h : S^{2Nm-2k-1} \to E^m\backslash\{0\};\ h \text{ is continuous and } S^1\text{-equivariant}\} ,$$

$$A_k^m = \{A \subset E^m\backslash\{0\};\ A = h(S^{2Nm-2k-1}),\ h \in \mathcal{H}_k^m\} .$$

This family of sets allows one to construct critical values for I^* on E^m by a mini-max type principle. We define

$$c_k^m = \sup_{A\in A_k^m} \min_{x\in A} I^*(x) , \tag{3.1}$$

for all m, $k \in \mathbb{N}^*$, $m > k + 1$.

Some properties of these numbers are listed in the next propositions.

<u>Proposition 3.1.</u> Suppose $V \in C^1(\mathbb{R}^N,\mathbb{R})$ satisfies *(V)* and $V(0) = 0$. Then:

i) $0 < c_k^m \leq c_{k+1}^m < +\infty \quad \forall\, m,k \in \mathbb{N}^*,\ m > k + 2$

ii) For all $k \in \mathbb{N}^*$, there exists $\mu(k)$ and $\nu(k)$ such that

$0 < \mu(k) \leq c_k^m \leq \nu(k) < +\infty \quad \forall\, m > k + 1.$

iii) Moreover, $\lim_{k\to+\infty} \mu(k) = +\infty$.

<u>Proposition 3.2.</u> For any $k \in \mathbb{N}^*$ such that $\mu(k) > 0$, c_k^m is a critical value of the restriction of I^* to E^m. Furthermore, the limit of any convergent subsequence of c_k^m as $m \to +\infty$ is a critical value of I^*.

Before proving these propositions, let us observe that, as a corollary, one derives from them the following result of Benci [7] and Rabinowitz [20, 23] concerning the periodic solutions of the autonomous system (1.5).

Theorem 3. Suppose $V \in C^1(\mathbb{R}^N,\mathbb{R})$ satisfies (V). Then, the autonomous system (1.5) possesses at least one non-constant T-periodic solution for any $T > 0$.

Proof of Theorem 3. We actually derive here a slightly stronger version of this result. We show that for any $A > 0$, there exists a non-constant periodic solution x of (1.4) such that $\|x\|_{L^\infty} > A$. Indeed, let

$$c_k = \overline{\lim_{m\to+\infty}}\, c_k^m .$$

By Propositions 3.1 and 3.2, $0 < c_k < \infty$, and $\lim_{k\to+\infty} c_k > \lim_{k\to+\infty} \mu(k) = +\infty$. Furthermore, c_k is a critical value of I^* (as soon as $\mu(k) > 0$).

Now let x_0 be a constant function, i.e. $x_0 \in E^0$. Then

$$I^*(x_0) = -2\pi V(x_0) < 2\pi b$$

for it follows from (1.2) that $-V(x) < b$, $\forall x \in \mathbb{R}^N$. Thus, I^* is bounded from above on E^0 and therefore, for large k, c_k corresponds to a non-constant periodic solution of (1.5). Let x_k denote a critical point of I^* associated with c_k: $x_k \in E$, $I^*(x_k) = c_k$, $(I^*)'(x_k) = 0$. We claim that $\|x_k\|_{L^\infty} \to +\infty$ as $k \to +\infty$. Indeed, arguing by way of contradiction let us assume that $\|x_k\|_{L^\infty}$ remains bounded along a subsequence. Since from the equation one derives that

$$\int_0^{2\pi} |\dot{x}_k|^2 dt = \int_0^{2\pi} V'(x_k)\cdot x_k dt ,$$

it is straightforward to see that $I^*(x_k)$ would then also remain bounded. This being impossible, the proof is thereby complete. ∎

We now turn to the proofs of the propositions.

Proof of Proposition 3.1. This result parallels Proposition 3.3 in [6]. The proof of i) which is quite simple (and identical to that in [6]) is omitted here. Let us prove ii). Consider the functional

$$J(x) = \frac{1}{2}\int_0^{2\pi} |\dot{x}|^2 - \frac{a}{p+1}\int_0^{2\pi} |x|^{p+1} . \tag{3.2}$$

(From now on, the measure dt is understood in all integrals over $[0,2\pi]$). Define

(3.3) $$d_k^m = \sup_{A \in A_k^m} \min_{x \in A} J(x) .$$

By (1.2), one has

(3.4) $$c_k^m \leq d_k^m + 2\pi b .$$

We require the following intersection lemma. It's proof is a straightforward adaptation from [6, Lemma 3.1] and will be omitted here. (It is a consequence from a version of Borsuk's theorem for the S^1-action; see [6]) and the references therein).

<u>Lemma 3.1.</u> For any $A \in A_{Nk}^m$, one has $A \cap E^{k+1} \neq \phi$.

Let us now show the existence of $\nu(k) < \infty$ such that $c_k^m \leq \nu(k)$, $\forall\, m \geq k+1$. Since $c_k^m \leq c_{k+1}^m$, it suffices to prove that for each k and $m \geq Nk+1$, c_{Nk}^m is bounded from above (by $\nu(Nk)$) independently of m. From Lemma 3.1 it follows that

(3.5) $$\min_{x \in A} J(x) \leq \max_{x \in E^{k+1}} J(x) , \quad \forall\, A \in A_{Nk}^m .$$

Now, for $x \in E^{k+1}$, one has

$$\int_0^{2\pi} |\dot{x}|^2 \leq (k+1)^2 \int_0^{2\pi} |x|^2 .$$

Therefore,

(3.6) $$J(x) \leq \frac{(k+1)^2}{2} \int_0^{2\pi} |x|^2 - \frac{a}{p+1} \int_0^{2\pi} |x|^{p+1}, \quad \forall\, x \in E^{k+1} .$$

Since the right hand side of (3.6) is obviously bounded from above independently of $x \in E^{k+1}$, we conclude, using (3.3) - (3.6) that

$$c_k^m \leq \nu(k) < +\infty \quad \forall\, k,m \in \mathbb{N}^*, \quad m \geq k+1 .$$

We now turn to the lower bound $\mu(k)$ for the c_k^m. We construct an explicit set $A \in A_k^m$ in the same way as in [6]. Incidentally, this will also show that $A_k^m \neq \phi$ whence that the c_k^m are well defined. Let $k = Nq - \ell$ with $q, \ell \in \mathbb{N}$, $0 \leq \ell < N$. $\mathbb{C}^\ell$ is identified to a subspace of $\mathbb{C}^N$ in the usual way. For $\beta = (\rho_1 e^{i\theta_1}, \ldots, \rho_N e^{i\theta_N}) \in \mathbb{C}^N$ and

$j \in \mathbb{N}$, $(\rho_1,\dots,\rho_N \in \mathbb{R}_+$, $\theta_1,\dots,\theta_N \in \mathbb{R}/2\pi\mathbb{Z})$, we denote

$$\beta^{(j)} = (\rho_1 e^{ij\theta_1},\dots,\rho_N e^{ij\theta_N}) .$$

Write $\zeta \in S^{2Nm-2k-1}$ as $\zeta = (\zeta_q,\zeta_{q+1},\dots,\zeta_m)$ with $\zeta_j \in \mathbb{C}^N$ for $q+1 \leq j \leq N$, $\zeta_q \in \mathbb{C}^\ell \subset \mathbb{C}^N$, and $\sum_{j=q}^m |\zeta_j|^2 = 1$. We define a mapping $h = S^{2Nm-2k-1} \to E^m \setminus \{0\}$ by setting:

$$h(\zeta)(t) = \frac{1}{\sqrt{2\pi}} \sum_{j=q}^m \frac{1}{j}\, \zeta_j^{(j)} e^{ijt} + \frac{1}{j}\, (\bar{\zeta}_j)^{(j)} e^{-ijt} . \tag{3.7}$$

Let $E_q^m = E^m \cap (E^{q-1})^\perp$. Then, $h : S^{2Nm-2k-1} \to E_q^m \setminus \{0\} \subset E^m \setminus \{0\}$. Indeed, one checks that for any $y \in h(S^{2Nm-2k-1})$ one has $\int_0^{2\pi} |\dot{y}|^2 = 1$. Furthermore, h is continuous, and h verifies:

$$h(e^{i\tau}\zeta)(t) = h(\zeta)(t+\tau), \qquad \forall\, e^{i\tau} \in S^1, \ \forall\, t \in \mathbb{R} .$$

(Just observe that $(e^{i\tau}\beta)^{(j)} = e^{ij\tau}(\beta)^{(j)}$). That is, h is equivariant under the S^1-action. Thus, $h \in \mathcal{K}_k^m$ and $A_k^m \neq \emptyset$: c_k^m is well defined.

Consider the mapping $\tilde{h}(\zeta) = \dfrac{h(\zeta)}{\|h(\zeta)\|_{L^\infty}}$. Then, again $\tilde{h} : S^{2Nm-2k-1} \to E_q^m \setminus \{0\} \subset E^m \setminus \{0\}$ is continuous. Since $\|T_\tau x\|_{L^\infty} = \|x\|_{L^\infty}$ $\forall\, \tau \in \mathbb{R}/2\pi\mathbb{Z}$, it is clear that $\tilde{h}$ is equivariant. Hence $\tilde{h} \in \mathcal{K}_k^m$, and $A = \tilde{h}(S^{2Nm-2k-1}) \in A_k^m$. For any $x \in A$, one has $\|x\|_{L^\infty} = 1$. To conclude, we require the following simple lemma.

<u>Lemma 3.2.</u> For any $x \in (E^{q-1})^\perp$, one has

$$\|x\|_{L^\infty} \leq \frac{1}{\sqrt{k_0 - 1}} \frac{1}{\sqrt{\pi}} \|\dot{x}\|_{L^2}$$

<u>Proof of Lemma 3.2.</u> Let $x \in (E^{q-1})^\perp$; x has a Fourier series expansion

$$x = \sum_{\substack{|j| \geq q \\ j \in \mathbb{Z}}} a_j e^{ijt}, \qquad a_j \in \mathbb{C}^N, \ a_{-j} = \bar{a}_j$$

One has

$$\|\dot{x}\|^2_{L^2} = 2\pi \sum_{|j| \geq q} |a_j|^2 j^2 ,$$

and

(3.8)
$$\|x\|_{L^\infty} \leq \sum_{|j| \geq q} |a_j| .$$

Writing $|a_j| = (|a_j| |j|) j^{-1}$, one derives from (3.8):

(3.9)
$$\|x\|_{L^\infty} \leq \Big\{ \sum_{|j| \geq q} |a_j|^2 j^2 \Big\}^{1/2} \Big\{ \sum_{|j| \geq q} |j|^{-2} \Big\}^{1/2} .$$

That is,

(3.10)
$$\|x\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}} \|\dot{x}\|_{L^2} \sqrt{2} \Big\{ \sum_{j \geq q} j^{-2} \Big\}^{1/2}$$

and the Lemma follows.

We now conclude the proof of Proposition 3.1. Let $A = \tilde{h}(S^{2Nm-2k-1})$. Then, by the definition (3.1) of c_k^m, one has

(3.11)
$$c_k^m \geq \min_{x \in A} I^*(x) .$$

Since $A \subset \{x \in E_q^m ,\ \|x\|_{L^\infty} = 1\} \subset S_q$, with $S_q = \{x \in (E^{q-1})^\perp ;\ \|x\|_{L^\infty} = 1\}$, one derives from (3.11):

(3.12)
$$c_k^m \geq \inf_{x \in S_q} I^*(x), \qquad \forall\, m \geq k+1 .$$

Hence, in particular, there exists for each k some $x_k \in S_q$ such that

(3.13)
$$c_k^m \geq I^*(x_k) - 1 = \mu(k), \qquad \forall\, m \geq k+1 .$$

We claim that $\lim_{k \to +\infty} \mu(k) = +\infty$. Indeed, since $\|x_k\|_{L^\infty} = 1$, $\int_0^{2\pi} V(x_k)$ is bounded independently of k. By Lemma 3.2, on the other hand, one has

$$\int_0^{2\pi} |\dot{x}_k|^2 \geq \pi(q-1) .$$

When $k \to +\infty$ one also has $q \to +\infty$, whence $\int_0^{2\pi} |\dot{x}_k|^2 \to +\infty$ and $I^*(x_k) \to +\infty$.

The proof of Proposition 3.1 is thereby complete. ∎

Proof of Proposition 3.2. We only sketch the proof here as it is essentially classical. To begin with, we observe that I^* satisfies the following Palais-Smale condition:

$$\text{(P.S)} \qquad \begin{cases} \text{For any sequence } (x_n) \subset E \text{ such that} \\ I^*(x_n) \le C \text{ and } (I^*)'(x_n) \to 0 \text{ in } E', \\ \text{then } (x_n) \text{ is relatively compact in } E. \end{cases}$$

Here and thereafter, C denotes various positive constants. The restriction of I^* to E^m, $I^*|_{E^m}$, satisfies the analogous property in E^m:

$$\text{(P.S)}_m \qquad \begin{cases} \forall (x_n) \subset E^m \text{ such that } I^*(x_n) \le C \text{ and} \\ (I^*|_{E^m})'(x_n) \to 0 \text{ in } (E^m)', \text{ then} \\ (x_n) \text{ is relatively compact in } E^m. \end{cases}$$

The proofs of these properties relying on condition (V) are by now classical and we shall not repeat them here. (See e.g. Rabinowitz [22] and Bahri-Berestycki [3, 6] for the derivation of these properties in related situations).

That c_k^m is a critical value of $I^*|_{E^m}$ as soon as $\mu(k) > 0$ follows from the definition of c_k^m and the property $(P.S)_m$ for $I^*|_{E^m}$. One can indeed adapt the type of argument given e.g. in Rabinowitz [22] to the present framework. The only modification which is required with respect to [22] concerns the "deformation lemma". Here one needs an appropriate "deformation" in the space E^m which, in addition to the usual properties, is equivariant under the S^1-action on E^m. The proof of this fact is but an adaptation from the argument in [22] and is left to the reader(1). A more general "equivariant deformation lemma" for the action of a compact Lie group is given in Benci [8] and could be used as well here. Lastly, let us just remark that the hypothesis $\mu(k) > 0$ is imposed because a set A in A_k^m is required to be included in $E^m \setminus \{0\}$. Thus, one has to construct the

(1) If I^* is of class C^2, one does not require this equivariant deformation lemma since one can work directly with the gradient flow of I^* which indeed is equivariant.

proper deformation of A which leaves 0 invariant. This is possible if one a priori knows that $c_k^m > 0$.

Let $k \in \mathbb{N}^*$ be such that $\mu(k) > 0$. Since $0 < \mu(k) \leq c_k^m \leq \nu(k) < +\infty$, the sequence $(c_k^m)_{m \geq k+1}$ possesses a convergent subsequence when $m \to +\infty$. Let m_j be a sequence of integers, $m_j \geq k+1$, such that $m_j \to +\infty$ and $c_k^{m_j} \to c_k \in \mathbb{R}$; then, $0 < \mu(k) \leq c_k \leq \nu(k) < +\infty$. For $m = m_j$, since c_k^m is a critical value of $I^*|_{E^m}$, there exists $x_m \in E^m$ with

$$I^*(x_m) = c_k^m \qquad (I^*|_{E^m})'(x_m) = 0 \ . \tag{3.14}$$

Let P^m denote the orthogonal projection of E onto E^m, $(m = m_j)$. One then has:

$$\frac{1}{2}\int_0^{2\pi} |\dot{x}_m|^2 - \int_0^{2\pi} V(x_m) \leq C \tag{3.15}$$

and

$$-\ddot{x}_m = P^m V'(x_m) \ . \tag{3.16}$$

Multiplying (3.16) by x_m and integrating yields:

$$\int_0^{2\pi} |\dot{x}_m|^2 = \int_0^{2\pi} V'(x_m)\cdot x_m \ . \tag{3.17}$$

Using (V) it is straightforward to derive from (3.15) and (3.17) that $\int_0^{2\pi} |\dot{x}_m|^2$, $\int_0^{2\pi} V(x_m)$ and $\int_0^{2\pi} V'(x_m)\cdot x_m$ are bounded independently of $m = m_j$. Using (1.2), one derives that $\int_0^{2\pi} |x_m|^{p+1}$ is bounded too and so is $\|x_m\|_E$. Therefore, one can strike out from (x_m) a further subsequence, denoted again by (x_m) such that $x_m \to x$ weakly in E, $x_m \to x$ strongly in L^∞, and $P^m V'(x_m) \to V'(x)$ strongly in L^2 whence in E'. Using (3.16) we conclude that $x_m \to x$ strongly in E. Clearly, x is a critical point of I^* and $I^*(x) = \lim\limits_{m=m_j \to +\infty} c_k^m = c_k$. Thus, c_k is a critical value of I^*.

This completes the proof of Proposition 3.2. ∎

To conclude this section, we recall from [6] a topological property of the level sets of I^* associated with the numbers c_k^m. This property is the key to the perturbative method for proving Theorem 1 which is developed in Section 6. Throughout the remaining of the paper we use the following notations. For a functional $\phi : E \to \mathbb{R}$, for $a \in \mathbb{R}$ and $m \in \mathbb{N}^*$, we denote (interchangeably):

$$[\phi]_a = [\phi > a] = \{x \in E;\ \phi(x) > a\}$$

$$[\phi]_a^m = [\phi > a]^m = \{x \in E^m;\ \phi(x) > a\} .$$

Theorem 4. Suppose that for some $\varepsilon > 0$ and some m, $k \in \mathbb{N}^*$, one has $0 < c_{k-1}^m + \varepsilon < c_k^m - \varepsilon$. Then, for any set $W \subset E^m$ such that

$$[I^* > c_{k-1}^m + \varepsilon]^m \supset W \supset [I^* > c_k^m - \varepsilon]^m$$

one has

$$\pi_{2Nm-2k-1}(W,x_0) \neq 0, \quad \text{for some } x_0 \in W .$$

Proof of Theorem 4. As it is quite simple, we repeat here the argument from [6, Theorem 3]. We argue by contradiction and suppose that $\pi_{2Nm-2k-1}(W,\cdot) = 0$. By the definition of c_k^m, there exists $h : S^{2Nm-2k-1} \to E^m\backslash\{0\}$ which is continuous, S^1-equivariant and such that

$$h(S^{2Nm-2k-1}) \subset [I^* > c_k^m - \varepsilon]^m \subset W$$

Since $\pi_{2Nm-2k-1}(W,\cdot) = 0$, there exists a homotopy

$$U : [0,1]\times S^{2Nm-2k-1} \to W$$

such that

$$\left.\begin{array}{l} U(0,\zeta) = h(\zeta) \\ U(1,\zeta) = x_0 \end{array}\right. \quad \forall\ \zeta \in S^{2Nm-2k-1} .$$

Write

$$S^{2Nm-2k+1} = \{\xi = (\zeta,\rho e^{i\theta});\ \zeta \in \mathbb{C}^{Nm-k},\ \rho \in \mathbb{R}_+ ,\ \theta \in \mathbb{R}/2\pi\mathbb{Z},\ |\zeta|^2 + \rho^2 = 1\} .$$

Now define $\tilde{h} : S^{2Nm-2k+1} \to E^m\backslash\{0\}$ by setting:

$$\tilde{h}(\zeta,\rho e^{i\theta}) = \begin{cases} h(\zeta) & \text{if } \rho = 0, \ |\zeta| = 1 , \\ T_\theta U(\rho, e^{-i\theta} \frac{\zeta}{|\zeta|}) & \text{if } \rho \neq 0, \ \zeta \neq 0 , \\ T_\theta x_0 & \text{if } \rho = 1, \ \zeta = 0 . \end{cases}$$

Then, it is easily checked that $\tilde{h}$ is continuous and S^1-equivariant. Since I^* is invariant under the S^1-action, the level sets of I^* are invariant sets under this action. Therefore, as $U(t,\zeta) \in W \subset [I^* > c_{k-1}^m + \varepsilon]^m$, one has

$$\tilde{h}(S^{2Nm-2k+1}) \subset [I^* > c_{k-1}^m + \varepsilon]^m \tag{3.18}$$

This implies in particular that $0 \notin \tilde{h}(S^{2Nm-2k+1})$. Thus, $\tilde{h} \in \mathcal{K}_{k-1}^m$ and (3.18) reads:

$$\min_{x \in \tilde{h}(S^{2Nm-2k+1})} I^*(x) > c_{k-1}^m + \varepsilon$$

which contradicts the very definition of c_{k-1}^m. The proof of Theorem 4 is thereby complete. ∎

4. A DETAILED STUDY OF SOME AUTONOMOUS EQUATION

In order to apply the preceding theorem, it is crucial, as will be seen in Section 6, to have a sharp estimate from below on the growth of the critical values c_k as $k \to +\infty$. Such an estimate will be derived in the next section. Some preliminary results are first required that we prove in the present section. They concern the precise description and some qualitative properties of the solutions to some auxiliary autonomous equation.

Consider the problem:

$$-\ddot{v} = g(v) \qquad (v(t) \in \mathbb{R}) \tag{4.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a given function. Throughout this section, g will be assumed to satisfy the following properties:

$$g : \mathbb{R} \to \mathbb{R} \text{ is of class } C^1, \text{ is odd and } g(0) = g'(0) = 0 . \tag{4.2}$$

$$g \text{ is increasing and convex on } [0,+\infty) \tag{4.3}$$

(4.4) $$0 < G(t) = \int_0^t g(s)ds < \theta g(t)t, \quad \forall\, t \neq 0 \quad \text{with} \quad 0 < \theta < \frac{1}{2}\,.$$

Let $\mathcal{E} = H^1(S^1)$; here $\mathcal{E}$ consists of scalar functions (note that $E = \mathcal{E}^N$). For $m \in \mathbb{N}^*$, consider the subspace of truncated Fourier series:

$$\mathcal{E}^m = \{x \in \mathcal{E};\ x = \sum_{j=-m}^{+m} a_j e^{ijt},\ a_j \in \mathbb{C},\ a_{-j} = \bar{a}_j,\ -m \le j \le m\}\,.$$

The next result provides a complete description of the set of 2π-periodic solutions of (4.1).

<u>Proposition 4.1.</u> Suppose g satisfies (4.2)-(4.4). There exists a sequence of nontrivial 2π-periodic solutions $(u_k)_{k\in\mathbb{N}^*}$ of (4.1) such that $u_k(0) = u_k(2\pi) = 0$. For each $k \in \mathbb{N}^*$, u_k is characterized by the properties that u_k has $2k-1$ zeros in $(0,2\pi)$ (all the zeros of u_k are simple) and $u_k'(0) > 0$. Furthermore, for any nontrivial solution v of (4.1), there exist $k \in \mathbb{N}^*$ and $\tau \in \mathbb{R}/2\pi\mathbb{Z}$ such that $v = T_\tau u_k$.

<u>Proof of Proposition 4.1.</u> Consider the nonlinear Sturm-Liouville problem:

(4.5) $$\begin{cases} -\ddot{w} = g(w) & \text{in } (0,2\pi)\,, \\ w(0) = w(2\pi) = 0\,. \end{cases}$$

It is known (see H. Berestycki [9]) that (4.5) exactly possesses a sequence of pairs of nontrivial solutions $\pm w_1, \pm w_2, \ldots, \pm w_j, \ldots$. For all j, w_j is characterized by the properties that $w_j'(0) > 0$ and w_j has $j-1$ zeros in $(0,2\pi)$, all of which are simple ("nodes"). Furthermore, these $\{\pm w_j\}_{j\in\mathbb{N}^*}$ together with $w_0 \equiv 0$ constitute <u>all</u> the solutions of (4.5) (see [5]). A simple integration by parts show that any solution w of (4.5) satisfies $(w'(2\pi))^2 - (w'(0))^2 = 0$, that is $w'(2\pi) = \pm w'(0)$. Hence, w_j is a periodic solution of (4.1) if and only if j is even: $j = 2k$, $k \in \mathbb{N}^*$. We denote $u_k = w_{2k}$, $\forall\, k \in \mathbb{N}$. Then, for any $k \in \mathbb{N}^*$ and $\tau \in \mathbb{R}/2\pi\mathbb{Z}$, $T_\tau u_k$ is a 2π-periodic solution of (4.1). We claim that 0 and $\{T_\tau u_k;\ k \in \mathbb{N}^*,\ \tau \in \mathbb{R}/2\pi\mathbb{Z}\}$ are the <u>only</u> 2π-periodic solutions of (4.1).

Indeed, let v be a non-constant 2π-periodic solution of (4.1); then $v \not\equiv 0$. There exists $\tau \in [0,2\pi]$ such that $v(\tau) = 0$. For if not, v would not change sign in $[0,2\pi]$. But this is impossible since by integrating (4.1), one sees that v satisfies:

$$\int_0^{2\pi} g(v) = 0$$

and $g(v)$ has the sign of v. Now, let $u = T_{-\tau}v$; u is a 2π-periodic solution of (4.5) and $u \not\equiv 0$. Hence, there exists $k \in \mathbb{N}^*$ such that $u = \pm u_k$. As it is easily checked, one has $-u_k = T_\pi u_k$. Therefore, either $v = T_\tau u_k$ or $v = T_{\tau+\pi}u_k$.

The proof of Proposition 4.1 is thereby complete. ∎

Let $G(z) = \int_0^z g(s)ds$ and consider the functional associated with (4.1):

$$\phi(v) = \frac{1}{2}\int_0^{2\pi} \dot{v}^2 - \int_0^{2\pi} G(v), \qquad v \in \mathcal{E} .$$

ϕ is a functional of class C^2 on $\mathcal{E}$ and

$$\langle \phi''(v)h,h\rangle = \int_0^{2\pi} \dot{h}^2 - \int_0^{2\pi} g'(v)h^2 .$$

(Recall that $\mathcal{E} \hookrightarrow L^\infty$). The critical points of ϕ on $\mathcal{E}$ are the 2π-periodic solutions of (4.1). Thus, the critical values of the functional ϕ on $\mathcal{E}$ are exactly the numbers

(4.6) $$\gamma_k = \phi(u_k), \qquad k \in \mathbb{N} .$$

(Notice that $\phi(T_\tau u_k) = \phi(u_k)$ $\forall\ \tau \in \mathbb{R}/2\pi\mathbb{Z}$). Our next result concerning (4.1) is an asymptotic property of the sequence γ_k as $k \to +\infty$.

<u>Proposition 4.2.</u> The sequence of critical values of ϕ satisfies the property

$$\lim_{k\to+\infty} \gamma_k/k^2 = +\infty$$

Furthermore, one has $0 < \gamma_1 < \gamma_2 < \dots < \gamma_k < \dots$.

<u>Proof of Proposition 4.2.</u> Let $v_k(t) = u_k(t/k)$. Then v_k is a 2π-periodic function. It is easily seen by symmetry properties that $u_k(2\pi/k) = 0$ and thus v_k is a solution of

$$(4.7)\qquad \begin{cases} -\ddot{v}_k = \dfrac{1}{k^2}\, g(v_k) \\ v_k(0) = v_k(2\pi) = 0\ . \end{cases}$$

(Recall that g is odd). One has $\int_0^{2\pi} \dot{u}_k^2 = k^2 \int_0^{2\pi} \dot{v}_k^2$ and $\int_0^{2\pi} G(u_k) = \int_0^{2\pi} G(v_k)$.
Hence

$$(4.8)\qquad \gamma_k/k^2 = \phi(u_k)/k^2 = \frac{1}{2}\int_0^{2\pi} \dot{v}_k^2 - \frac{1}{k^2}\int_0^{2\pi} G(v_k)\ .$$

(4.7) yields:

$$(4.9)\qquad \int_0^{2\pi} \dot{v}_k^2 = \frac{1}{k^2}\int_0^{2\pi} g(v_k)v_k\ .$$

Hence, one derives from (4.4), (4.8) and (4.9):

$$(4.10)\qquad \gamma_k/k^2 \geq \left(\frac{1}{2} - \theta\right)\int_0^{2\pi} \dot{v}_k^2\ .$$

We claim that $\int_0^{2\pi} \dot{v}_k^2 \to +\infty$ as $k \to +\infty$. Indeed, suppose by way of contradiction that for a subsequence of indices k, $\|\dot{v}_k\|_{L^2}$ remains bounded. Then, $\|v_k\|_{H_0^1}$ and consequently $\|v_k\|_{L^\infty}$ remain bounded. Hence, there exists a constant $C > 0$, independent of k such that $|g(v_k)| \leq C|v_k|$. By (4.9) this leads to

$$(4.11)\qquad 0 < \int_0^{2\pi} v_k^2 \leq 4\int_0^{2\pi} \dot{v}_k^2 \leq 4\,\frac{C}{k^2}\int_0^{2\pi} v_k^2$$

which is impossible for large k. Therefore, $\int_0^{2\pi} \dot{v}_k^2 \to +\infty$ as $k \to +\infty$, and from (4.10) it follows that

$$(4.12)\qquad \lim_{k\to+\infty} \gamma_k/k^2 = +\infty\ .$$

Let us now check that $\{\gamma_k\}_{k\in\mathbb{N}^*}$ is an increasing sequence. Actually, we are going to derive a stronger property. Namely, that $\{\gamma_k/k^2\}_{k\in\mathbb{N}^*}$ is an increasing sequence of positive numbers. For $\lambda > 0$, let w_λ be the unique solution of

$$(4.13)\qquad \begin{cases} -\ddot{w}_\lambda = \lambda g(w_\lambda), & w_\lambda > 0 \quad \text{in} \quad (0,\pi)\ , \\ w_\lambda(0) = w_\lambda(\pi) = 0 \end{cases}$$

It is proved in H. Berestycki [9] that w_λ exists and is unique. Moreover, owing to the a priori estimate derived in [9] and which can easily be adapted to (4.13), one verifies that $\lambda \mapsto w_\lambda$ is a C^1 mapping from $(0,+\infty)$ into $H_0^1((0,\pi))$. (Notice that this a priori estimate breaks down as $\lambda \downarrow 0$). Let

$$e(\lambda) = \frac{1}{2}\int_0^\pi \dot{w}_\lambda^2 - \lambda \int_0^\pi G(w_\lambda)\ .$$

Then,

$$\frac{de(\lambda)}{d\lambda} = \int_0^\pi \dot{w}_\lambda \frac{d\dot{w}_\lambda}{d\lambda} - \lambda \int_0^\pi g(w_\lambda)\frac{dw_\lambda}{d\lambda} - \int_0^\pi G(w_\lambda)\ .$$

But since $\frac{dw_\lambda}{d\lambda} \in H_0^1((0,\pi))$, one obtains from the equation (4.13):

$$\int_0^\pi \dot{w}_\lambda \frac{d\dot{w}_\lambda}{d\lambda} - \lambda \int_0^\pi g(w_\lambda)\frac{dw_\lambda}{d\lambda} = 0\ .$$

Hence, using the fact that $G(s) > 0 \quad \forall\, s \neq 0$, one has

$$(4.14)\qquad \frac{de(\lambda)}{d\lambda} < 0\ .$$

That is, $e(\lambda)$ is decreasing with respect to λ. Now, from (4.13) we derive the following expression of $e(\lambda)$:

$$e(\lambda) = \lambda \int_0^\pi \left\{\frac{1}{2}\, g(w_\lambda)w_\lambda - G(w_\lambda)\right\}\ .$$

Whence, by (4.4) we see that $e(\lambda) > 0$, $\forall\ \lambda > 0$.

Using the same notation as for the proof of the first part of the proposition, we know that v_k is positive on $(0,\pi)$ (as $u_k > 0$ on $(0,\pi/k)$) and $v_k(0) = v_k(\pi) = 0$. Therefore, from (4.7) it follows that v_k is the solution of (4.13) corresponding to $\lambda = 1/k^2$: $v_k = w_{k^{-2}}$. Thus, by (4.8), one has

(4.15) $$\gamma_k/k^2 = \frac{1}{2}\int_0^{2\pi} \dot{v}_k^2 - \frac{1}{k^2}\int_0^{2\pi} G(v_k) = 2e(1/k^2) .$$

Hence, by (4.14) we obtain that γ_k/k^2 is an increasing sequence of positive numbers.

This completes the proof of Proposition 4.2. ∎

Remark 4.1. In the particular case $g(s) = |s|^{q-1}s$ with $1 < q < \infty$, computations can be made somewhat more explicit. Indeed, it is easy to see in this case that all the u_k are deduced from u_1 by the transformation $u_k(t) = k^{2/(q-1)}u_1(kt)$. It then follows that the critical values of the functional $\psi(u) = \frac{1}{2}\int_0^{2\pi} \dot{u}^2 - \frac{1}{q+1}\int_0^{2\pi} |u|^{q+1}$ on $\mathcal{E}$ are the numbers $\bar{\gamma}_k = \psi(u_1)(k)^{2\frac{q+1}{q-1}}$, $k \in \mathbb{N}^*$ with $\psi(u_1) = (\frac{1}{2} - \frac{1}{q+1})\int_0^{2\pi} |u_1|^{q+1} > 0$. Since the exponent of k in $\bar{\gamma}_k$ may be made as close to 2 as one wishes, this example shows the result of Proposition 4.2 to be optimal. ∎

In order to use Theorem 2 in the next section, we now require a lower bound on the maximal dimension of a subspace of $\mathcal{E}^m$ on which the quadratic form $h \mapsto \langle \phi''(v)h,h\rangle$ is positive definite, when $v = T_\tau u_k$. This is the purpose of the next results.

Proposition 4.3. For each critical point $T_\tau u_k$ of ϕ, there exists a subspace F of $\mathcal{E}$ (depending on τ and k), F having codimension $2k + 1$, and there exists $\varepsilon > 0$ (ε depending on k) such that

$$\langle \phi''(T_\tau u_k)h,h\rangle \geq \varepsilon \|h\|^2_{L^2} , \qquad \forall\, h \in F .$$

Proof of Proposition 4.3. For a function $q \in L^\infty([0,2\pi])$, we let $\mu_1(q) < \dots < \mu_j(q) < \dots$ denote the sequence of eigenvalues of the linear Sturm-Liouville problem:

(4.16) $$\begin{cases} -\ddot{w} - qw = \mu w & \text{in } (0,2\pi) \\ w(0) = w(2\pi) = 0 \end{cases}$$

By a result of H. Berestycki [9], we know that the solution w_j of (4.5) has the property that

$$\mu_j(g'(w_j)) < 0 < \mu_{j+1}(g'(w_j)) .$$

Hence, in particular,

$$\mu_{2k}(g'(u_k)) < 0 < \mu_{2k+1}(g'(u_k)) . \tag{4.17}$$

Let $\{z_j\}_{j\in\mathbb{N}^*}$ denote the sequence (depending on k) of normalized eigenfunctions of (4.16) associated with $q = g'(u_k)$:

$$\begin{cases} -\ddot{z}_j - g'(u_k)z_j = \mu_j(g'(u_k))z_j & \text{in } (0,2\pi) \\ z_j(0) = z_j(2\pi) = 0, \quad z_j'(0) > 0, \quad \|z_j\|_{L^2} = 1 . \end{cases} \tag{4.18}$$

Consider the space $F_k = \text{span}\{z_j;\ j \geq 2k + 1\}$. Then, F is a subspace of $H_0^1((0,2\pi))$ having codimension $2k$ in $H_0^1((0,2\pi))$. Furthermore, because of (4.17), one obviously has

$$\langle \phi''(u_k)h,h\rangle \geq \epsilon_k\|h\|^2_{L^2} , \qquad \forall\, h \in F_k \tag{4.19}$$

where $\epsilon_k = \mu_{2k+1}(g'(u_k)) > 0$. For any function $w \in H_0^1((0,2\pi))$, one has $w(0) = w(2\pi) = 0$. Hence, one can identify $H_0^1((0,2\pi))$ to a subspace of $H^1(S^1) = \mathcal{E}$ and one has

$$\mathcal{E} = H_0^1((0,2\pi)) \oplus \mathbb{R} .$$

Therefore, F_k is a subspace of $\mathcal{E}$ of codimension $2k + 1$.

Now, for $\tau \in \mathbb{R}/2\pi\mathbb{Z}$, let $F_{k,\tau} = \{T_\tau h;\ h \in F_k\} = T_\tau F_k$. Obviously, F_τ is a subsapce of $\mathcal{E}$ having codimension $2k + 1$. An easy calculation shows that $F = F_{k,\tau}$ and $\epsilon = \epsilon_k > 0$ verify the desired properties in Proposition 4.3.

The proof of Proposition 4.3 is thereby complete. ∎

A straightforward corollary of Proposition 4.3 is the following:

<u>Corollary 4.1.</u> For any $m,k \in \mathbb{N}^*$, $m \geq k + 1$, and for any $\tau \in \mathbb{R}/2\pi\mathbb{Z}$, there exists a subspace F of $\mathcal{E}^m$ (F depends on m,k and τ) such that

$$\dim F \geq 2m - 2k - 1$$

and

$$\langle \phi''(T_\tau u_k)h,h\rangle \geq \varepsilon_k \|h\|^2_{L^2} \qquad \forall\, h \in F$$

for some $\varepsilon_k > 0$.

It just suffices to observe that if $\hat{F}$ is the subspace given by Proposition 4.2, then $F = \hat{F} \cap \mathcal{E}^m$ satisfies $\dim F \geq 2m - 2k - 1$. ∎

Remark 4.2. Define the coindex of a critical point v of ϕ with respect to $\mathcal{E}^m$, $\mathrm{coind}(v,\phi,\mathcal{E}^m)$, as the largest integer j such that there exists a subspace $H \subset \mathcal{E}^m$ having dimension j and such that

$$\langle \phi''(v)h,h\rangle > 0 \qquad \forall\, h \in H\setminus\{0\} .$$

Then, proposition 4.3 reads:

$$\mathrm{coind}(T_\tau u_k,\phi,\mathcal{E}^m) \geq 2m - 2k - 1 .$$ ∎

All the critical points of ϕ in $\mathcal{E}$ are given by the family $\{T_\tau u_k;\ k \in \mathbb{N},\ \tau \in \mathbb{R}/2\pi\mathbb{N}\}$. However, the critical points of the restriction $\phi_{|\mathcal{E}^m}$ of ϕ to the subspace $\mathcal{E}^m$ are different. Nevertheless, using the fact that the critical points of $\phi_{|\mathcal{E}^m}$ "approach" the critical points of ϕ in $\mathcal{E}$ when $m \to +\infty$, we will now derive a lower bound for the "coindex" of the critical points of $\phi_{|\mathcal{E}^m}$.

Proposition 4.4. Let $k \in \mathbb{N}^*$ and let $\delta \in \mathbb{R}$ be a number such that $\gamma_k < \delta < \gamma_{k+1}$. There exists an integer $m_0 = m_0(\delta) \in \mathbb{N}$ such that for any $m \geq m_0$, δ is not a critical value of $\phi_{|\mathcal{E}^m}$. Moreover, for any $v \in \mathcal{E}^m$ satisfying $\phi(v) < \delta$ and $(\phi_{|\mathcal{E}^m})'(v) = 0$, $m \geq m_0$, there exists a subspace $F \subset \mathcal{E}^m$ (F depends on v, m, δ) and there exists $\varepsilon > 0$ (depending only on δ) such that

$$\langle \phi''(v)h,h\rangle \geq \varepsilon \|h\|^2_{L^2} , \qquad \forall\, h \in F$$

and

$$\dim F \geq 2m - 2k - 1 .$$

Proof of Proposition 4.4. Let us first introduce some notations:

$$Z^\delta(\phi) = \{v \in \mathcal{E};\ \phi'(v) = 0,\ \phi(v) < \delta\}$$

$$Z^\delta_m(\phi) = \{v \in \mathcal{E}^m;\ (\phi_{|\mathcal{E}^m})'(v) = 0,\ \phi(v) < \delta\} .$$

For a set $A \subset E$ and a real $\alpha > 0$, we denote:

$$N_\alpha(A) = \{v \in E;\ \text{distance}(v,A) < \alpha\} .$$

The proof is divided into five steps.

Step 1. $Z^\delta(\phi)$ is a compact set in $\mathcal{E}$. This is a consequence of the Palais-Smale condition (P.S) satisfied by ϕ on $\mathcal{E}$.

Step 2. Let $(v_m) \subset E$ be a sequence defined for $m > m_1$ such that $v_m \in Z_m^\delta(\phi)$. Then (v_m) has a convergent subsequence which converges towards a point in $Z^\delta(\phi)$. (This is but a particular case of the proof given for Proposition 3.2 above).

Step 3. For any $\alpha > 0$, there exists $m_1 = m_1(\delta,\alpha) \in \mathbb{N}$ such that $\forall\, m > m_1$, one has $Z_m^\delta(\phi) \subset N_\alpha(Z^\delta(\phi))$. This fact is obtained arguing indirectly and using Step 2.

Step 4. For any $\varepsilon_1 > 0$, there exists $\eta > 0$ such that for any $v \in N_\eta(Z^\delta(\phi))$, one has for some $u \in Z^\delta(\phi)$:

$$(4.20) \qquad |\langle\phi''(v)h,h\rangle - \langle\phi''(u)h,h\rangle| < \varepsilon_1 \|h\|^2_{L^2}, \qquad \forall\, h \in \mathcal{E} .$$

This just follows from the C^2 character of the functional ϕ on $\mathcal{E}$ and from the fact that $Z^\delta(\phi)$ is compact.

Step 5. Conclusion: By proposition 4.3, there exists $\varepsilon > 0$, and for any $u \in Z^\delta(\phi)$ there exists a subspace F_u of $\mathcal{E}$ (F_u depending on u,δ) such that F_u has codimension $2k + 1$ and

$$(4.21) \qquad \langle\phi''(u)h,h\rangle > \varepsilon\|h\|^2_{L^2} \qquad \forall\, h \in F_u .$$

Let $\varepsilon_1 = \varepsilon/2 > 0$ and let $\eta > 0$ be defined by Step 4. Lastly, let $m_0 = m_1(\delta,\eta)$ be given by Step 3 (m_0 only depends on δ). Then, for any $m > m_0$ and any $v \in Z_m^\delta(\phi)$, there exists $u \in Z^\delta(\phi)$ such that (4.20) is verified. Whence it follows from (4.20) and (4.21) that

$$\langle\phi''(v)h,h\rangle > \frac{\varepsilon}{2}\|h\|^2_{L^2}, \qquad \forall\, h \in F_u \cap \mathcal{E}^m = \hat{F} .$$

Since $\dim \hat{F} > 2m - 2k - 1$, the proof of Proposition 4.4 is complete. ∎

To conclude this section, we consider now a functional Φ defined on $E = \mathcal{E}^N$ by

$$\Phi(x) = \sum_{i=1}^{N} \phi(x_i) = \frac{1}{2}\int_0^{2\pi} |\dot{x}|^2 - \int_0^{2\pi} W(x)$$

for any $x = (x_1,\ldots,x_N) : \mathbb{R} \to \mathbb{R}^N$, $x \in E$, and where $W(x) = \sum_{i=1}^{N} G(x_i)$. We denote here again

$$Z^\delta(\Phi) = \{x \in E;\ \Phi'(x) = 0,\ \Phi(x) < \delta\}$$

and

$$Z_m^\delta(\Phi) = \{x \in E^m;\ (\Phi_{|E^m})'(x) = 0,\ \Phi(x) < \delta\} .$$

From the above propositions, we obtain the following result for Φ.

<u>Proposition 4.5.</u> The critical values of Φ on E are the numbers $\beta_{k_1,\ldots,k_N} = \gamma_{k_1} + \gamma_{k_2} + \ldots + \gamma_{k_N}$ for any combination of integers $k_1,\ldots,k_N \in \mathbb{N}$, where γ_k is the k^{th} critical value of ϕ on $\mathcal{E}$ (see (4.6)). Let $\delta \in \mathbb{R}$ be a regular value of Φ. Define the integer $L(\delta)$ to be the largest sum $k_1 + \ldots + k_N$ among the N-uples $k_1,\ldots,k_N \in \mathbb{N}$ which satisfy $\beta_{k_1,\ldots,k_N} < \delta$. Then, there exists $m_0 = m_0(\delta) \in \mathbb{N}$ such that $\forall\, m > m_0$, δ is not a critical value of the restriction $\Phi_{|E^m}$. Moreover, for any $x \in Z_m^\delta(\Phi)$ with $m > m_0$, there exists a subspace F of E^m (F depending on x, m and δ) such that

$$\langle \Phi''(x)h,h\rangle > 0 \quad \forall\, h \in F\backslash\{0\}$$

and

$$\dim F > 2Nm - 2L(\delta) - N .$$

<u>Proof of Proposition 4.5.</u> For any $x = (x_1,\ldots,x_N) \in E$ and $h = (h_1,\ldots,h_N) \in E$, one has

$$\Phi'(x)h = \sum_{i=1}^{N} \phi'(x_i)h_i \tag{4.22}$$

$$\langle \Phi''(x)h,h\rangle = \sum_{i=1}^{N} \langle \phi''(x_i)h_i,h_i\rangle . \tag{4.23}$$

Hence, $\Phi'(x) = 0$ is equivalent to $\phi'(x_i) = 0$, $\forall\ i = 1,\dots,N$. Thus, the critical values of Φ are the numbers $\beta_{k_1,\dots,k_N} = \gamma_{k_1} + \dots + \gamma_{k_N}$. Let $\{e_1,\dots,e_N\}$ denote the canonical basis of $\mathbb{R}^N$. Since $E^m = \mathcal{E}^m e_1 \oplus \dots \oplus \mathcal{E}^m e_N$, it is also easily verified that

(4.24) $$\left(\Phi_{|E^m}\right)'(x) = 0 \Leftrightarrow \left(\phi_{|\mathcal{E}^m}\right)'(x_i) = 0, \quad \forall\ i = 1,\dots,N\ .$$

Let $x \in E$ be a critical point of Φ. Then, we know that $x = \left(T_{\tau_1}u_{k_1},\dots,T_{\tau_N}u_{k_N}\right)$ for some $\tau_1,\dots,\tau_N \in \mathbb{R}/2\pi\mathbb{Z}$ and $k_1,\dots,k_N \in \mathbb{N}$. By proposition 4.3, we know that there exist N subspaces of $\mathcal{E}$, $F_1,\dots,F_N$, with F_j having codimension $2k_j + 1$ in $\mathcal{E}$, and there exists $\varepsilon > 0$ such that

(4.25) $$\langle \phi''\left(T_{\tau_j}u_{k_j}\right)h_j,h_j\rangle \geq \varepsilon\, \|h_j\|^2_{L^2}\ , \quad \forall\ h_j \in F_j\ .$$

Moreover, an inspection of the proof of Proposition 4.3 shows at once that ε can be chosen independently of k_j provided each k_j is bounded from above by some $k \in \mathbb{N}$; ε then only depends on k. Let us assume henceforth that $\beta_{k_1,\dots,k_N} < \delta$. Then, for each k_j one has $k_j \leq L(\delta)$; and therefore, ε can be chosen to only depend on δ. Let $F = F_1e_1 \oplus \dots \oplus F_Ne_N$; F is a subspace of E having codimension $2(k_1 + \dots + k_N) + N$, and F depends on x and δ. By (4.23) and (4.25), one has

(4.26) $$\langle \Phi''(x)h,h\rangle \geq \varepsilon\|h\|^2_{L^2}, \quad \forall\ h \in F\ .$$

Now, to conclude the proof of Proposition 4.5 it just suffices to repeat the steps 1 to 5 in the proof of Proposition 4.4. Firstly, it is straightforward to check that Φ satisfies the Palais-Smale condition (P.S) in E. Therefore, $Z^\delta(\Phi)$ is compact and one shows that $\exists\, m_0 = m_0(\delta) \in \mathbb{N}$ such that for $m > m_0$, δ is not a critical value of $\Phi_{|E^m}$. Using the facts that $Z^\delta(\Phi)$ is compact and Φ is of class C^2 on E, one proves that there exists $\eta > 0$ such that for any $y \in N_\eta(Z^\delta(\Phi))$ one can find $x \in Z^\delta(\Phi)$ such that

(4.27) $$|\langle \Phi''(y)h,h\rangle - \langle \Phi''(x)h,h\rangle| < \frac{\varepsilon}{2}\, \|h\|^2_{L^2}\ , \quad \forall\ h \in E\ .$$

Lastly, following the same type of argument as the one used for Proposition 3.2 one shows that if $m_0(\delta) \in \mathbb{N}$ is large enough, then one has

(4.28) $$Z_m^\delta(\Phi) \subset N_\eta(Z^\delta(\Phi)), \qquad \forall\, m > m_0 .$$

The proof of Proposition 4.5 is completed by combining the inequalities (4.26) and (4.27). These show that for any $m > m_0$ and for any $y \in Z_m^\delta(\Phi)$ there exists a subspace F of E^m such that

$$\langle \Phi''(y)h,h\rangle > 0 \qquad \forall\, h \in F\setminus\{0\}$$

and

$$\dim F > 2Nm - 2(k_1 + \ldots + k_N) - N > 2Nm - 2L(\delta) - N . \qquad ∎$$

5. AN ESTIMATE FROM BELOW ON THE GROWTH OF THE CRITICAL VALUES

The results of the preceding sections will enable us to derive here a sharp estimate from below on the growth of the critical values of I^* constructed in Section 3. The main result of this section is the following.

Theorem 5. Suppose $V \in C^1(\mathbb{R}^N,\mathbb{R})$ satisfies condition (V). Let c_k^m be the critical values of $I^*|_{E^m}$ defined by (3.1) and let $c_k = \overline{\lim\limits_{m\to+\infty}}\, c_k^m$, $(0 < c_k < \infty)$. There exists a subsequence c_{k_i} $(k_i \to +\infty$ as $i \to +\infty)$ such that

$$\lim_{k_i\to+\infty} c_{k_i}/k_i^2 = +\infty .$$

In the proof of this Theorem, we require the following technical lemma.

Lemma 5.1. Let $V \in C^0(\mathbb{R}^N,\mathbb{R})$ be an arbitrarily given function. There exists a function $G \in C^2(\mathbb{R},\mathbb{R})$ having the following properties

(5.1) $G' = g$ is odd

(5.2) $G(0) = g(0) = g'(0) = 0$

(5.3) g is increasing and convex on $[0,+\infty)$

(5.4) $0 < G(s) < \frac{1}{3}\, g(s)s, \quad \forall\, s \in \mathbb{R},\ s \neq 0 .$

(5.5) $$V(x) < \sum_{i=1}^{N} G(x_i) + C \qquad \forall\, x = (x_1,\ldots,x_N) \in \mathbb{R}^N$$

where C is a constant.

Proof of Lemma 5.1. Set

$$m_n = \max_{\substack{|x|\leq n \\ x\in\mathbb{R}^N}} V(x) .$$

Choose a sequence of positive numbers $a_0, a_1, \ldots, a_m$, such that $a_n > 0 \ \forall n \in \mathbb{N}$, and

$$a_0 > m_1$$

$$a_0 + a_1 > m_2$$

$$------$$

$$\sum_{i=1}^{n} a_i > m_{n+1} .$$

Define for $r \in \mathbb{R}$, $r \geq 0$:

$$g_1(r) = 3 \sum_{n=1}^{+\infty} a_n[(r - n + 1)^+]^2$$

where c^+ stands for $\max(c,0)$. Observe that g_1 is a finite sum for any $r \in \mathbb{R}_+$. Clearly, $g_1 \in C^1(\mathbb{R}_+,\mathbb{R})$ and

$$G_1(r) = \int_0^r g_1(s)ds$$

verifies $G_1(0) = g_1(0) = g_1'(0) = 0$. Moreover, g_1 is increasing and strictly convex on $[0,+\infty)$ and one has

$$G_1(r) = \sum_{n=1}^{+\infty} a_n[(r - n + 1)^+]^3 \leq \sum_{n=1}^{+\infty} a_n[(r - n + 1)^+]^2 r = \frac{1}{3} g_1(r)r, \quad \forall r \geq 0 .$$

Lastly, one has

(5.6) $$a_0 + G_1(n) \geq a_0 + a_1 + \ldots + a_n \geq m_{n+1}$$

Now define for $r \geq 0$:

$$g(r) = \sqrt{N}\, g_1(\sqrt{N}\, r)$$

$$G(r) = \int_0^r g(s)ds = G_1(\sqrt{N}\, r) .$$

For $r \in \mathbb{R}$, $r < 0$, set $g(-r) = -g(r)$ and $G(r) = G(-r)$. It is obvious to check that G satisfies properties (5.1)-(5.4). Let $x \in \mathbb{R}^N$ and let $n \in \mathbb{N}$ be such that $n \leq |x| \leq n + 1$. Hence, $V(x) \leq m_{n+1}$ and there exists $j \in \{1,\ldots,N\}$ such that

$\sqrt{N}\,|x_j| \geq n$. Therefore, since $G \geq 0$ and G is increasing on $\mathbb{R}_+$, we obtain:

$$a_0 + \sum_{i=1}^{N} G(x_i) \geq a_0 + G(x_j) = a_0 + G_1(\sqrt{N}\,|x_j|) \geq a_0 + G_1(n) .$$

Hence, using (5.6), we derive:

$$a_0 + \sum_{i=1}^{N} G(x_i) \geq V(x) \qquad \forall\, x = (x_1,\ldots,x_N) \in \mathbb{R}^N ,$$

that is, property (5.5). This concludes the proof of Lemma 5.1. ∎

<u>Proof of Theorem 5.</u> We use here the notations of Sections 3 and 4. Let G be the function given by Lemma 5.1. Define

$$\phi(v) = \frac{1}{2}\int_0^{2\pi} \dot{v}^2 - \int_0^{2\pi} G(v) \qquad \forall\, v \in \mathcal{E}$$

and

$$\Phi(x) = \sum_{i=1}^{N} \phi(x_i) = \frac{1}{2}\int_0^{2\pi} |\dot{x}|^2 - \int_0^{2\pi} W(x) ,$$

for $x = (x_1,\ldots,x_N) \in E$, where $W(x) = \sum_{i=1}^{N} G(x_i)$. For $m,k \in \mathbb{N}^*$, $m \geq k+1$, define

$$b_k^m = \sup_{A \in A_k^m} \min_{x \in A} \Phi(x)$$

and

$$b_k = \overline{\lim_{m\to+\infty}}\, b_k^m , \qquad \forall\, k \in \mathbb{N}^* .$$

Since $\Phi - 2\pi a_0 \leq I^*$ (by (5.5)), one has

(5.7) $$b_k^m - 2\pi a_0 \leq c_k^m , \qquad \forall\, m,k \in \mathbb{N}^*,\ m \geq k+1$$

and

(5.8) $$b_k - 2\pi a_0 \leq c_k , \qquad \forall\, k \in \mathbb{N}^* .$$

Thus, to establish Theorem 5, it suffices to show that there exists a subsequence b_{k_i} of (b_k) ($k_i \to +\infty$ as $i \to +\infty$) such that

(5.9) $$\lim_{k_i\to+\infty} b_{k_i}/k_i^2 = +\infty .$$

The functional Φ is a particular case of the class of functionals studied in Section 3. Indeed, $W \in C^2(\mathbb{R}^N,\mathbb{R})$ and W verifies condition (V) (with $\theta = \frac{1}{3}$). Hence, all the results of Section 3 apply to Φ, and we know that

(5.10) $$\lim_{k\to\infty} b_k = +\infty$$

(5.11) $$b_k \leq b_{k+1} \quad \forall\, k \in \mathbb{N}^*$$

(5.12) $$b_k \text{ is a critical value of } \Phi \quad \forall\, k \in \mathbb{N}^* .$$

(Notice that 0 is a critical value of Φ since $\Phi'(0) = 0$). We also recall that Theorem 3 applies here with I^* and c_k^m replaced by Φ and b_k^m respectively.

By (5.12) and Proposition 4.5, we know that for any $k \in \mathbb{N}$, there exist N integers $j_1,\dots,j_N \in \mathbb{N}$ such that

$$b_k = \beta_{j_1,\dots,j_N} = \gamma_{j_1} + \gamma_{j_2} + \dots + \gamma_{j_N} ,$$

where the γ_j, $j \in \mathbb{N}$, are defined in (4.6). By (5.10) and (5.11), there exists a subsequence b_{k_i} of (b_k), with $k_i \to +\infty$ as $i \to +\infty$, such that

(5.13) $$b_{k_i-1} < b_{k_i}, \quad \forall\, i \in \mathbb{N} .$$

We claim that (5.13) implies (5.9). This fact rests on the following lemma.

Lemma 5.2. For any $k \in \mathbb{N}$, $k \geq 2$ such that $b_{k-1} < b_k$, there exists $j_1,\dots,j_N \in \mathbb{N}$ with $j_1 + \dots + j_N + N \geq k$, and $\gamma_{j_1} + \dots + \gamma_{j_N} < b_k$.

Proof of Lemma 5.2. We argue by contradiction and suppose that for any $\beta_{j_1,\dots,j_N} < b_k$, one has $j_1 + \dots + j_N < k - N$. There exists $\delta \in \mathbb{R}$, $b_{k-1} < \delta < b_k$ such that $[\delta,b_k)$ does not contain any critical value of Φ. As in Proposition 4.5, define $L(\delta)$ to be the largest sum $j_1 + \dots + j_N$ among the N-uples of integers $j_1,\dots,j_N \in \mathbb{N}$ subject to the constraint $\beta_{j_1,\dots,j_N} < \delta$. Then, one has

(5.14) $$2L(\delta) + N < 2k - N < 2k .$$

By Proposition 4.5, we know that there exists $m_0 = m_0(\delta) \in \mathbb{N}^*$ such that δ is not a critical value of $\Phi_{|E^m}$, for any $m \geq m_0$. Furthermore, for any $m \geq m_0$ and for any $x \in Z_m^\delta(\Phi) = \{x \in E^m, (\Phi_{|E^m})'(x) = 0, \Phi(x) \leq \delta\}$, there exists a subspace F_x of E^m (F_x depends on x, m and δ) such that

(5.15) $$\dim F \geq 2Nm - 2L(\delta) - N$$

and

(5.16) $$\langle \Phi''(x)h,h\rangle > 0 \quad \forall\, h \in F_x \backslash \{0\}\ .$$

Lastly, from the proof of Proposition 4.5 it is straightforward to derive that $Z_m^\delta(\Phi)$ is compact.

Hence, we are now in a position to apply Theorem 2 of Section 2 to obtain, using (5.15):

(5.17) $$\pi_\ell([\Phi]_\delta^m, p) = 0, \ \forall\, \ell \in \mathbb{N}^*,\ \ell \leq (2Nm - 2L(\delta) - N) - 1, \ \forall\, p \in [\Phi]_\delta^m\ .$$

By (5.14) we have $2Nm - 2k - 1 \leq (2Nm - 2L(\delta) - N) - 1$. Therefore, (5.17) yields:

(5.18) $$\pi_{2Nm-2k-1}([\Phi]_\delta^m, p) = 0, \ \forall\, m \geq m_0, \ \forall\, p \in [\Phi]_j^m\ .$$

On the other hand, there exists an $m \in \mathbb{N}^*$ large enough, with $m \geq m_0$, and such that

(5.19) $$b_{k-1}^m < \delta < b_k^m\ .$$

Then, by Theorem 3 of Section 3, the inequalities (5.19) imply:

(5.20) $$\pi_{2Nm-2k-1}([\Phi]_\delta^m, p) \neq 0$$

for some $p \in [\Phi]_\delta^m$.

The contradiction between (5.18) and (5.20) completes the proof of Lemma 5.2. ∎

<u>Conclusion of the proof of Theorem 5.</u> A consequence of Lemma 5.2 is that for any $k \in \mathbb{N}^*$, $k \geq 2N$, such that $b_{k-1} < b_k$, there exists $j \in \mathbb{N}$ with $j \geq \frac{1}{2N} k$ and $\gamma_j \leq b_k$. (Indeed, the γ_j's are positive, and if $j_1 + \ldots + j_N \geq k - N$, at least one j_i verifies $j_i \geq \frac{1}{2N} k$).

Now let (k_i) be the subsequence satisfying (5.13) $(k_i \to +\infty)$. Then, for any k_i, there exists $j_i \in \mathbb{N}$ such that

(5.21) $$\gamma_{j_i} \leq b_{k_i},\ j_i \geq \frac{1}{2N} k_i\ .$$

Hence, $j_i \to +\infty$ as $i \to +\infty$. From (5.21) we derive

$$b_{k_i}/k_i^2 \geq \frac{1}{(2N)^2} \gamma_{j_i}/(j_i)^2 . \tag{5.22}$$

Therefore, by Proposition 4.2, we derive from (5.22):

$$\lim_{k_i \to +\infty} b_{k_i}/k_i^2 = +\infty .$$

The proof of Theorem 5 is thereby complete. ∎

Remark 5.1. If one assumes V to have polynomial growth, that is $V(x) \leq a'|x|^{q+1} + b'$ for some $a',b' > 0$ and $q > 1$, then the above estimate can be somewhat sharpened. As is clear from the proof above, one can show in this case, using Remark 4.1 that for a subsequence c_{k_i}, one has

$$c_{k_i} \geq \mu (k_i)^{2 \frac{q+1}{q-1}}$$

where $\mu > 0$ is some constant. This result will be used in Section 7. ∎

Remark 5.2. We conjecture that one actually has $\lim_{k \to +\infty} c_k/k^2 = +\infty$ for the whole sequence $(c_k)_{k \in \mathbb{N}}$. The estimate of Theorem 5 was derived here using some deep topological properties associated with the numbers c_k. It would be interesting to know if one can derive this estimate (or a stronger version) in a purely analytical fashion. Lastly, another open problem is to know whether one can achieve a more precise understanding of the relationship between the integers $k \in \mathbb{N}$ and $j_1,\dots,j_N \in \mathbb{N}$ which satisfy $b_k = \beta_{j_1,\dots,j_N}$. We emphasize the fact that even in the simple case $N = 1$ and $V(x) = \frac{1}{q+1}|x|^{q+1}$ $(q > 1)$, such a relation or such a stronger estimate for the whole sequence b_k are not yet known. ∎

Theorem 5 will be used in the next section through its following corollary.

Lemma 5.3. Let $V \in C^1(\mathbb{R}^N,\mathbb{R})$ satisfy condition (V). Let $A > 0$, $k > 0$, $\sigma_1,\sigma_2 > 0$ be arbitrarily given positive numbers and let $p > 1$ be given. There exist $k \in \mathbb{N}^*$, $k \geq 2$ and a sequence $(m_j) \subset \mathbb{N}^*$, $m_j \to +\infty$ as $j \to +\infty$ such that $k \geq K$ and for $m = m_j$ the following hold:

$$\lim_{m=m_j\to+\infty} c_k^m = \overline{\lim_{m\to+\infty}}\, c_k^m = c_k$$

$$A \leq c_{k-1}^m < c_k^m$$

and

$$c_k^m - c_{k-1}^m \geq \sigma_1 (c_k^m)^{1/(p+1)} + \sigma_2$$

for all indices $m = m_j$.

Proof of Lemma 5.3. Let $c_k = \overline{\lim_{m\to+\infty}}\, c_k^m$, $\forall\, k \in \mathbb{N}^*$. It clearly suffices to show that there exists $k \in \mathbb{N}$, $k \geq 2$ such that $k \geq K$ with

$$c_{k-1} \geq \mu(k-1) > A$$

and

(5.23) $$c_k - c_{k-1} > \sigma_1 (c_k)^{1/(p+1)} + \sigma_2 .$$

We claim that (5.23) holds for an infinite sequence of indices $k \in \mathbb{N}^*$. (This is enough to conclude since $\lim_{k\to+\infty} \mu(k) = +\infty$). We argue by contradiction and suppose that

(5.24) $$c_k - c_{k-1} \leq \sigma_1 (c_k)^{1/(p+1)} + \sigma_2, \quad \forall\, k \geq k_0$$

for some $k_0 \in \mathbb{N}^*$. Using a slight modification of Lemma 5.1 in [3] (or Lemma 7.5 in [6]), it is straightforward to show that (5.24) implies

(5.25) $$c_k \leq \alpha\, k^{(p+1)/p} + \beta, \quad \forall\, k \in \mathbb{N}^*$$

for some constants $\alpha > 0$, $\beta \geq 0$. Since $p > 1$, $(p+1)/p < 2$ and (5.25) yields

$$\lim_{k\to+\infty} c_k/k^2 = 0 .$$

But this is impossible as it would contradict the result in Theorem 5. The proof of the lemma is thereby complete.

6. EXISTENCE OF FORCED OSCILLATIONS

Using the results of the previous sections we will now prove Theorem 1. Recall that I is the functional defined by

$$I(x) = \frac{1}{2}\int_0^{2\pi} |\dot{x}|^2 - \int_0^{2\pi} V(x) + \int_0^{2\pi} f\cdot x\ , \qquad x \in E$$

The critical points of I in E are the 2π-periodic solutions of the system

$$\ddot{x} + V'(x) = f(t)\ . \tag{1.1}$$

We start by a truncation procedure on the functional I.

Let $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ be a C^∞ function with the following properties:

$$\chi(s) = 1\ , \quad \forall\ s \in [0,1]\ ,$$

$$\chi(s) = 0\ , \quad \forall\ s > 2\ ,$$

$$\chi'(s) < 0\ , \quad \forall\ s \in \mathbb{R}_+\ .$$

For $\rho > 1$, set $\tilde{\chi}_\rho(s) = \chi(s/\rho)$. Thus, $\tilde{\chi}_\rho$ verifies

$$\tilde{\chi}_\rho \in C^\infty(\mathbb{R}_+,\mathbb{R}_+),\quad 0 < \tilde{\chi}_\rho < 1,\quad \chi'_\rho < 0 \quad \text{on } \mathbb{R}_+\ , \tag{6.1}$$

$$\tilde{\chi}_\rho(s) = 1 \quad \forall\ s \in [0,\rho] \quad \text{and} \quad \tilde{\chi}_\rho(s) = 0 \quad \forall\ s > 2\rho\ , \tag{6.2}$$

$$|\tilde{\chi}'_\rho(s)\cdot s| < B \quad \forall\ s > 0 \tag{6.3}$$

where $B > 0$ is a constant. Lastly, we set

$$\chi_\rho(x) = \tilde{\chi}_\rho\Big(\int_0^{2\pi} |x|^{p+1}\Big) \qquad \forall\ x \in E\ , \tag{6.4}$$

where p is the exponent appearing in (1.2). (E.g. $p + 1 = 1/\theta$ with θ given by condition (V) is admissible in (1.2)).

For $\rho > 1$, we define

$$I_\rho(x) = \frac{1}{2}\int_0^{2\pi} |\dot{x}|^2 - \int_0^{2\pi} V(x) + \chi_\rho(x)\int_0^{2\pi} f\cdot x\ .$$

Thus, if $\|x\|^{p+1}_{L^{p+1}} < \rho$, one has $I_\rho = I$ in an L^{p+1}-neighborhood of x in E, while if $\|x\|^{p+1}_{L^{p+1}} > 2\rho$, then $I_\rho = I^*$ in an L^{p+1} neighborhood of x in E.

We require the next three technical lemmas.

Lemma 6.1. $|I^*(x) - I_\rho(x)| < \mu\, \rho^{1/(p+1)}$, $\forall\ x \in E$, where $\mu > 0$ is a constant.

<u>Proof of Lemma 6.1.</u> One has

$$|I^*(x) - I_\rho(x)| \leq C\ \chi_\rho(x)\ \|x\|_{L^{p+1}} .$$

The lemma follows from the fact that $\chi_\rho(x) = 0$ as soon as $\|x\|_{L^{p+1}} > (2\rho)^{1/(p+1)}$. ∎

<u>Lemma 6.2.</u> For any $\rho > 1$, I_ρ satisfies the Palais-Smale condition:

$$\text{(P.S)} \quad \begin{cases} \text{For any sequence } (x_j) \subset E \text{ such that } I_\rho(x_j) \text{ is bounded from above and} \\ (I_\rho)'(x_j) \to 0 \text{ strongly in } E', \text{ then } (x_j) \text{ is relatively compact in } E. \end{cases}$$

Furthermore, $I_{\rho|_{E^m}}$ satisfies the analogous property in E^m for all $m \in \mathbb{N}^*$. Lastly, I_ρ verifies the condition.

$$\text{(P.S)}^* \quad \begin{cases} \text{For any sequence } (x_m) \subset E \text{ such that } x_m \in E^m,\ (I_{\rho|_{E^m}})'(x_m) = 0 \text{ and such that} \\ I_\rho(x_m) \text{ is bounded from above, there exists a convergent subsequence from } (x_m) \\ \text{which converges to a critical point of } I_\rho. \end{cases}$$

The proof of Lemma 6.2 is essentially classical. It uses property (6.3) and it relies on arguments that have already been called previously in this paper. It is also straightforward to adapt the Appendix in [6] to the present framework to derive this lemma. Lastly, one could also adapt the estimates in the proof of the next lemma in order to obtain Lemma 6.2. We therefore omit the details here.

<u>Lemma 6.3.</u> There exist two constants $\alpha > 0$ and $\beta > 0$ such that for any $\rho > 1$ one has the following property. If $x \in E$ verifies $(I_\rho)'(x) = 0$ and $I_\rho(x) \leq \alpha\rho - \beta$, then $\|x\|^{p+1}_{L^{p+1}} \leq \rho - 1$ and consequently, $I_\rho = I$ in a neighborhood of x in E.

<u>Proof of Lemma 6.3.</u> $(I_\rho)'(x) = 0$ reads

$$\ddot{x} + V'(x) = \chi_\rho(x)\ f + \chi'_\rho(x) \int_0^{2\pi} f \cdot x \ , \tag{6.5}$$

where

$$\chi'_\rho(x) = (p + 1)\ \tilde{\chi}'_\rho(\|x\|^{p+1}_{L^{p+1}})\ |x|^{p-1}x \ .$$

Hence

$$\langle \chi'_\rho(x), x\rangle = (p+1)\ \tilde{\chi}'_\rho(\|x\|^{p+1}_{L^{p+1}})\ \|x\|^{p+1}_{L^{p+1}} .$$

Therefore, by (6.3) one has

(6.6) $$|\langle \chi'_\rho(x), x\rangle| \le B_1 = (p+1)B .$$

Multiplying (6.5) by x and integrating yields:

(6.7) $$\left|\int_0^{2\pi} |\dot{x}|^2 - \int_0^{2\pi} V'(x)\cdot x\right| \le C\|x\|_{L^{p+1}} ,$$

where we have used (6.6), (6.1) and where $C > 0$ denotes a constant - as it continues to do generically in the sequel.

Now, in addition to (6.5) suppose that one has

(6.8) $$I_\rho(x) \le A$$

for some $A > 0$. Then, using (6.7), (6.8) and condition (V) one derives

(6.9) $$\int_0^{2\pi} V(x) \le CA + C + C\|x\|_{L^{p+1}} .$$

Using (1.2), one obtains from (6.9) that

(6.10) $$\int_0^{2\pi} |x|^{p+1} \le CA + C' .$$

Let us choose $\alpha, \beta > 0$ in such a way that $C\alpha \le 1$ and $-C\beta + C' \le -1$, where C and C' are the positive constants displayed in (6.10). We have thus shown that $(I_\rho)'(x) = 0$ and $I_\rho(x) \le \alpha\rho - \beta$ together imply the estimate $\|x\|^{p+1}_{L^{p+1}} \le \rho - 1$. Notice that α and β do not depend on ρ. ∎

We also require the next corollary:

<u>Lemma 6.4.</u> Let α and β be the constants of Lemma 6.3. For any $\rho > 1$, there exists $m_0(\rho) \in \mathbb{N}^*$ such that for any $m \ge m_0(\rho)$ one has the following property: If $x \in E^m$ verifies $(I_{\rho|E^m})'(x) = 0$ and $I_\rho(x) \le \alpha\rho - \beta$, then $I_\rho = I$ in a neighborhood of x in E^m.

This follows easily from Lemmas 6.3 and 6.4. ∎

To prove Theorem 1, we will now show that I has a sequence of critical values which is unbounded from above. We argue by contradiction and suppose that the critical values of I are bounded from above. That is, we make the following assumption.

$$(6.11)\qquad \begin{cases} \text{There exists } A \subset R \text{ such that } I \\ \text{has no critical values in } [A,+\infty). \end{cases}$$

Then, by Lemma 6.2 (condition (P.S)), the set of critical points of I, $Z(I)$ is compact. For any functional $F \in C^1(E,\mathbb{R})$, we continue to denote

$$Z^{\delta}(F) = \{x \in E;\ F'(x) = 0,\ F(x) \leq \delta\}$$

$$Z_m^{\delta}(F) = \{x \in E;\ (F_{|E^m})'(x) = 0,\ F(x) \leq \delta\}\ .$$

From (6.11) and Lemmas 6.2 and 6.4 we know that by choosing $m_0(\rho)$ large enough one has

$$(6.12)\qquad Z_m^{\alpha\rho-\beta}(I_\rho) \subset N_\eta(Z(I)),\quad \forall\ \rho > 1,\quad \forall\ m > m_0(\rho)$$

where, as usual, $N_\eta(Z(I)) = \{x \in E;\ \text{distance}\ (x,Z(I)) < \eta\}$ and where $\eta > 0$ is some fixed positive number (e.g. $\eta = 1$). Since $E \hookrightarrow L^\infty$, we derive from (6.12) that

$$(6.13)\qquad \begin{cases} \exists\ C > 0 \text{ such that } \|x\|_{L^\infty} \leq C \text{ for any} \\ x \in Z_m^{\alpha\rho-\beta}(I_\rho) \text{ and for any } m > m_0(\rho),\ \forall\ \rho > 1\ . \end{cases}$$

In (6.13), C is independent of ρ and m. Since $V \in C^2(\mathbb{R}^N,\mathbb{R})$, one obtains

$$(6.14)\qquad \|V''(x)\|_{L^\infty} \leq C,\quad \forall\ x \in Z_m^{\alpha\rho-\beta}(I_\rho),\quad \forall\ \rho > 1,\quad \forall\ m > m_0(\rho)\ .$$

The estimate (6.14) yields a lower bound on the <u>coindex</u> of the critical points of I_ρ. Indeed, let $j_0 \in \mathbb{N}$ be a fixed integer such that $j_0^2 > C$ (C is the constant in (6.14). One has

$$\langle I''(x)h,h\rangle = \int_0^{2\pi} |\dot{h}|^2 - \int_0^{2\pi} V''(x)h\cdot h\ .$$

Hence, for any $h \in E^m \cap (E^{j_0})^{\perp}\setminus\{0\}$ and for any $x \in E^m$ satisfying $\|V''(x)\|_{L^\infty} < C$, one has

(6.15) $$\langle I''(x)h,h\rangle \geq [(j_0+1)^2 - C]\int_0^{2\pi} h^2 > 0 \ .$$

By Lemma 6.4, we know that if $x \in Z_m^{\alpha\rho-\beta}(I_\rho)$ and $m \geq m_0(\rho)$, then $I_\rho = I$ in a neighborhood of x. Therefore,

(6.16) $$\langle I_\rho''(x)h,h\rangle = \langle I''(x)h,h\rangle \quad \forall\, h \in E^m \ .$$

We sum up (6.14) - (6.16) in the following relation:

(6.17) $$\begin{cases} \langle I_\rho''(x)h,h\rangle > 0, & \forall\, h \in E^m \cap (E^{j_0})^{\perp}\setminus\{0\} \ , \\ \forall\, x \in Z_m^{\delta}(I_\rho), & \forall\, m \geq m_0(\rho), \quad \forall\, \delta \in [A, \alpha\rho - \beta] \end{cases}$$

(Indeed, observe that by (6.11) and Lemmas 6.2 - 6.4, one has
$Z_m^{\alpha\rho-\beta}(I_\rho) = Z_m^{\delta}(I_\rho) = Z_m^{A}(I_\rho) \subset Z_m^{A}(I)$ for any $\delta \in [A, \alpha\rho - \beta]$ and any $m \geq m_0(\rho)$).

We can now apply Theorem 2 (see Section 2). Let ρ_0 be defined by $\alpha\rho_0 - \beta = A$. By assumption (6.11) and Lemmas 6.2-6.3 we know that if $\rho \geq \rho_0$, $[A, \alpha\rho - \beta]$ does not contain any critical value of I_ρ or of $I_{\rho}|_{E^m}$ provided $m \geq m_0(\rho)$. The relation (6.17) shows that for any $x \in Z_m^{\alpha\rho-\beta}(I_\rho)$ there exists a $2N(m - j_0)$-dimensional subspace of E^m on which I_ρ'' is positive definite. By Theorem 2 this implies:

(6.18) $$\begin{cases} \pi_\ell([I_\rho]_\delta^m) = 0, & \forall\, \ell \in \mathbb{N}^*, \quad \ell \leq 2N(m - j_0) - 2 \\ \forall\, \rho \geq \rho_0, & \forall\, m \geq m_0(\rho), \quad \forall\, \delta \in [A, \alpha\rho - \beta] \ . \end{cases}$$

We will now show that (6.18) to which (6.11) led is untenable. Firstly, in view of Lemma 6.1 notice that one has

(6.19) $$[I^*]_{b_1}^m \supset [I_\rho]_{b_2}^m \supset [I^*]_{b_3}^m$$

as soon as

(6.20) $$b_2 \geq b_1 + \mu\rho^{1/(p+1)} \quad \text{and} \quad b_3 \geq b_2 + \mu\rho^{1/(p+1)}$$

(where $\mu > 0$ is the constant given by Lemma 6.1). By Lemma 5.3, there exist $k \in \mathbb{N}^*$ and a sequence $(m_j) \subset \mathbb{N}^*$, $m_j \to +\infty$ such that for all $m = m_j$ the following hold:

(6.21) $$k \geq Nj_0 + 1 \ ,$$

(6.22) $$\underline{\lim}_{m=m_j\to+\infty} c_k^m = \overline{\lim}_{m\to+\infty} c_k^m = c_k ,$$

(6.23) $$\beta, \lambda < c_{k-1}^m < c_k^m ,$$

(6.24) $$c_k^m - c_{k-1}^m > \sigma_1 (c_k^m)^{1/(p+1)} + \sigma_2 ,$$

for all $m = m_j$, where $\sigma_1, \sigma_2 > 0$ are arbitrarily fixed positive numbers.

We precisely choose σ_1, σ_2 in such a way that one has

$$\sigma_1 a^{1/(p+1)} + \sigma_2 > 2\mu\left(\frac{a + \beta}{\alpha}\right)^{1/(p+1)} + 2 ,$$

for any $a > 0$, where α, β are given by Lemma 6.3 and $\mu > 0$ is the constant in Lemma 6.1. Inequality (6.24) then leads to

(6.25) $$c_k - c_{k-1} > 2\mu \left(\frac{c_k + \beta}{\alpha}\right)^{1/(p+1)} + 2 .$$

Let $\rho = \dfrac{c_k + \beta}{\alpha}$. We now fix $m = m_j$ large enough so that $m > m_0(\rho)$, $c_k^m - \frac{1}{2} < c_k$ and

(6.26) $$c_k^m - c_{k-1}^m > 2\mu\rho^{1/(p+1)} + 1 .$$

Set

(6.27) $$\delta = c_{k-1}^m + \frac{1}{2} + \mu\rho^{1/(p+1)}$$

Then, by (6.26) one obtains

(6.28) $$c_k^m - \frac{1}{2} > \delta + \mu\rho^{1/(p+1)} .$$

By Lemma 6.1 (compare with (6.19)-(6.20)) we have:

(6.29) $$\left[I^* > c_{k-1}^m + \frac{1}{2}\right]^m \supset [I_\rho > \delta]^m \supset \left[I^* > c_k^m - \frac{1}{2}\right]^m$$

Whence, by Theorem 4 (Section 3) one derives from (6.29) that

(6.30) $$\pi_{2Nm-2k-1}([I_\rho > \delta]^m, w) \neq 0$$

for some point $w \in [I_\rho]_\delta^m = [I_\rho > \delta]^m$. Observe now that

$$\lambda < c_{k-1}^m + \frac{1}{2} < \delta < c_k^m - \frac{1}{2} < c_k = \alpha\rho - \beta$$

and that $m > m_0(\rho)$. We have thus reached in (6.30) a contradiction to (6.18) for by (6.21) one knows that

$$2Nm - 2k - 1 \leq 2N(m - j_0) - 3 .$$

Thus, the assumption (6.11) is absurd and the proof of Theorem 1 is thereby complete.

Remark 6.1. In the preceding argument the assumption that V was C^2 played a crucial role in obtaining the bound (6.14), which allowed us to invoke Theorem 2. We would like to emphasize that a simpler argument allows one to prove the existence of at least one forced vibration of (1.1) (for any given periodic f) under the assumption that $V \in C^1(\mathbb{R}^N,\mathbb{R})$. Indeed, the above proof shows that (6.29), whence (6.30), hold for at least one $k \in \mathbb{N}$ and for an infinite sequence of $m = m_j \to +\infty$. Now suppose that (1.1) has no solutions at all. Then, $I_{\rho}|_{E^m}$ has no critical values in $(-\infty,\delta]$ for a fixed $m = m_j$ large enough. By Lemma 2.1 then, the set $[I_\rho]_\delta^m$ is a deformation retract of the whole space E^m. This implies $\pi_\ell([I_\rho]_\delta^m) = 0$, $\forall \ell \in \mathbb{N}^*$ which is a contradiction to (6.30).

Remark 6.2. It is easy to check that the contradiction of assumption (6.11) actually gives the following slightly stronger result: There exists a sequence $(x_k)_{k\in\mathbb{N}}$ of 2π-periodic solutions of (1.1) such that $\lim_{k\to+\infty} \|x_k\|_{L^\infty} = +\infty$ Note that $\|x_k\|_{L^\infty}$ is the amplitude of a 2π-periodic solution. ∎

7. MORE GENERAL FORCED SYSTEMS

In this section, we consider the more general non-autonomous system

(1.3) $$\ddot{x} + \hat{V}'_x(t,x) = 0 .$$

Here again, we are interested in the existence of T-periodic solutions $x(t) \in \mathbb{R}^N$ for (1.3). We assume that $\hat{V}$ satisfies

(7.1) $$\hat{V} \in C^2(\mathbb{R} \times \mathbb{R}^N,\mathbb{R}) \text{ and } \hat{V}(t,x) \text{ is T-periodic in } t .$$

(7.2) $$\begin{cases} 0 < V(t,x) \leq \theta\, V'_x(t,x)\cdot x \quad \forall\, x \in \mathbb{R}^N, \ |x| > R_0 \\ \text{where } \theta \in (0,1/2) . \end{cases}$$

(7.1)-(7.2) imply the existence of positive constants $\gamma,\delta > 0$ such that

(7.3) $$\gamma|x|^{p+1} - \delta \le \hat{V}(t,x) \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N ,$$

where $p + 1 = 1/\theta > 2$. Thus V is superquadratic in x.(1)

The results and methods of the previous sections allow us to show the following result for (1.3).

<u>Theorem 6.</u> Let $\hat{V} \in C^2(\mathbb{R} \times \mathbb{R}^N,\mathbb{R})$ verify (7.1) and (7.2). Suppose that there exists a function $V \in C^2(\mathbb{R}^N,\mathbb{R})$ satisfying condition (V) and such that

(7.4) $$|\hat{V}(t,x) - V(x)| \le C + C|x|^\alpha \quad \forall t \in \mathbb{R},\ \forall x \in \mathbb{R}^N ,$$

where $C > 0$ is a constant and $\alpha > 0$. If α is such that $\alpha < \frac{p+1}{2} = \frac{1}{2\theta}$, then, problem (1.3) possesses infinitely many T-periodic solutions.

<u>Remark 7.1.</u> $p + 1$ is the exponent appearing in the relation (1.2) satisfied by V. Note that from (V) one can choose $p + 1 = 1/\theta$. The number $\theta \in (0,1/2)$ is the <u>same</u> in (V) and in (7.2). ∎

<u>Remark 7.2.</u> (1.1) is a particular case of system (1.3) corresponding to $\hat{V}(t,x) = V(x) - f(t)\cdot x$. Since $\alpha = 1$ is always admissible in Theorem 6, one sees that (for $f \in L^\infty$) Theorem 6 is an extension of Theorem 1. ∎

<u>Sketch of the proof of Theorem 6.</u> Since the proof follows exactly the same ideas as the one we have developed above for Theorem 1, we just mention here the general outline and some estimates.

As before, we fix $T = 2\pi$ and observe that the 2π-periodic solutions of (1.3) are the critical points in E of the functional

$$J(x) = \frac{1}{2}\int_0^{2\pi} |\dot{x}|^2 - \int_0^{2\pi} \hat{V}(t,x) .$$

(1) The existence of <u>subharmonics</u> (that is kT periodic solutions of (1.3) with $k \in \mathbb{N}^*$) has been studied by P. H. Rabinowitz [21] for certain classes of Hamiltonian systems, different from the ones considered here. (For instance, in the case of (1.2) where $\hat{V}(t,x) = V(x) - f(t)\cdot x$, the hypotheses in [21] would imply $f \equiv 0$). For a <u>subquadratic</u> $\hat{V}$, the existence of subharmonics in (1.3) has also been proved by F. Clarke and I. Ekeland [27].

For $\rho > 1$, set

$$J_\rho(x) = \frac{1}{2}\int_0^{2\pi} |\dot{x}|^2 - \chi_\rho(x)\int_0^{2\pi} \hat{V}(t,x) - (1 - \chi_\rho(x))\int_0^{2\pi} V(x) ,$$

where, as in Section 6, $\chi_\rho(x)$ stands for

$$\chi_\rho(x) = \hat{\chi}_\rho\left(\int_0^{2\pi} |x|^{p+1}\right) ,$$

and χ_ρ, $\hat{\chi}_\rho$ verify (6.1)-(6.4).

The proof of Theorem 6 rests on the following estimates which parallel Lemmas 6.1-6.4.

Lemma 7.1. Under assumption (7.4), for $\alpha < p + 1$, one has $|J_\rho(x) - I^*(x)| < \mu\, \rho^{\alpha/(p+1)}$, $\forall\, x \in E$, $\forall\, \rho > 1$, where $\mu > 0$ is a constant.

Lemma 7.2. J_ρ and $J_\rho|_{E^m}$ verify the Palais-Smale condition in E and E^m respectively. Moreover, J_ρ satisfies the condition (P.S)*.

Lemma 7.3. There exist two constants $\alpha, \beta > 0$ such that for any $\rho > 1$ one has the following property. If $x \in E$ verifies $(J_\rho)'(x) = 0$ and $J_\rho(x) < \alpha\rho - \beta$, then $\|x\|^{p+1}_{L^{p+1}} < \rho - 1$ and consequently, $J_\rho = J$ in a neighborhood of x in E. Furthermore, Lemma 6.4 holds with I and I_ρ replaced by J and J_ρ respectively.

The proofs of these lemmas follow very closely a priori estimates already derived in this paper (see in particular Lemmas 6.1-6.4). We therefore do not repeat them here. ∎

From Lemma 7.1 it follows that

$$[I^*]^m_d \supset [J_\rho]^m_a \supset [I^*]^m_{d'} ,$$

provided $a > d + \mu\,\rho^{\alpha/(p+1)}$ and $d' > a + \mu\,\rho^{\alpha/(p+1)}$. Let c_k be the critical values of I^* defined by (3.1). Using the same method of proof as in Section 6, one can find a number a_k such that $c_{k-1} + \frac{1}{2} < a_k < c_k - \frac{1}{2}$ and

(7.5) $$\left[I^* > c^m_{k-1} + \tfrac{1}{2}\right]^m \supset [J_\rho > a_k]^m \supset \left[I^* > c^m_k - \tfrac{1}{2}\right]^m$$

for infinitely many indices m, if one has

(7.6) $$c_k - c_{k-1} > 2\mu\,\rho^{\alpha/(p+1)} + 2 .$$

Now, in view of Lemma 7.2, one furthermore requires that ρ, k be chosen in such a way that

(7.7) $$c_k < \alpha\rho - \beta ,$$

thereby insuring that $a_k < \alpha\rho - \beta$. The inequalities (7.6) and (7.7) are compatible (that is, one can find a $\rho > 1$ satisfying both) provided c_{k-1} and c_k verify

(7.8) $$c_k - c_{k-1} > \sigma_1 c_k^{\alpha/(p+1)} + \sigma_2$$

for some appropriate constants $\sigma_1, \sigma_2 > 0$.

Thus, for any $k \in \mathbb{N}^*$ such that (7.8) holds, there exists $\rho > 1$ and $a_k < \alpha\rho - \beta$ for which the inclusions (7.5) are valid for infinitely many indices m. By Theorem 4, this implies

(7.9) $$\pi_{2Nm-2k-1}([J_\rho > a_k]^m) .$$

We have seen in the preceding section that by Theorem 1, one derives from the fact that (7.9) holds for infinitely many indices k that J possesses a sequence of critical values which is unbounded from above. (This is obtained via an argument by contradiction).

Therefore, to prove Theorem 6, it suffices to show that (7.8) holds for infinitely many indices k. By way of contradiction suppose that

(7.10) $$c_k - c_{k-1} < \sigma_1 c_k^{\alpha/(p+1)} + \sigma_2$$

for any $k > k_0$. Then, by Lemma 5.1 in [3] on Lemma 7.5 in [6], there exists a constant $M > 0$ such that

(7.11) $$c_k < M\, k^{\frac{p+1}{p+1-\alpha}} , \quad \forall\, k > 1 .$$

By Theorem 5 (Section 5) there exists a sequence $(k_i) \subset \mathbb{N}^*$, $k_i \to +\infty$ such that

(7.12) $$\lim_{k_i \to +\infty} c_{k_i}/k_i^2 = +\infty .$$

Thus, one readily sees that (7.10) is impossible if $(p+1)[p+1-\alpha]^{-1} < 2$, that is if $\alpha < \frac{p+1}{2}$. Hence, in this case, (7.8) holds for infinitely many indices k and the proof of Theorem 6 is complete. ∎

As we have seen in Section 5, estimates on the growth of c_k sharper than (7.11) can be achieved under additional assumptions on V. More precisely, suppose V verifies

(7.13) $$a|x|^{p+1} - b \le V(x) \le a'|x|^{q+1} + b', \qquad \forall\, x \in \mathbb{R}^N,$$

with $a,b,a',b' > 0$ being constants and $1 < p \le q < \infty$. Then, we know (compare Remark 5.1) that there exists a sequence $(k_i) \subset \mathbb{N}$, $k_i \to +\infty$ and a constant $\nu > 0$ such that

(7.14) $$c_{k_i} \ge \nu\, k_i^{2\frac{q+1}{q-1}}$$

In this situation, (7.11) (which comes from contradicting (7.8)) is impossible provided

(7.15) $$\frac{p+1}{p+1-\alpha} < 2\,\frac{q+1}{q-1},$$

that is, $\alpha < \frac{(p+1)(q+3)}{2(q+1)}$.

We thus have shown:

Theorem 7. Let V and $\hat{V}$ verify the assumptions of Theorem 6. Suppose moreover that V satisfies (7.13). Then, the conclusion of Theorem 6 holds with $\alpha < \frac{(p+1)(q+3)}{2(q+1)}$. In particular, if $p = q$ in (7.13), then the conclusion holds with $\alpha < \frac{p+3}{2}$.

Remark 7.3. The preceding results lead quite naturally to an open problem: It is tempting to conjecture that (1.3) possesses infinitely many T-periodic solutions provided $\hat{V}$ only satisfies (7.1) and (7.2). ∎

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This article studies the existence of T-periodic solutions for systems of nonlinear second order ordinary differential equations of the type $\ddot{x} + V'(x) = f(t)$. Here, $x : \mathbb{R} \to \mathbb{R}^N$, $V \in C^2(\mathbb{R}^N,\mathbb{R})$ and $f : \mathbb{R} \to \mathbb{R}^N$ is a given T-periodic forcing term ($T > 0$ is given). Assuming V to be superquadratic, it is shown that this system possesses infinitely many T-

20. ABSTRACT (cont.)

periodic solutions. The proof of this result rests on showing that certain homotopy groups of level sets of the functional associated with the system are not trivial. Some more general results concerning systems of the type $\ddot{x} + \hat{V}_x'(t,x) = 0$ are also presented here.

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